

SURJECTIVITY OF GAUSSIAN MAPS FOR CURVES ON ENRIQUES SURFACES

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ABSTRACT. Making suitable generalizations of known results we prove some general facts about Gaussian maps. The above are then used, in the second part of the article, to give a set of conditions that insure the surjectivity of Gaussian maps for curves on Enriques surfaces. To do this we also solve a problem of independent interest: a tetragonal curve of genus $g \geq 7$ lying on an Enriques surface and general in its linear system, cannot be, in its canonical embedding, a quadric section of a surface of degree $g - 1$ in \mathbb{P}^{g-1} .

1. INTRODUCTION

Gaussian maps have emerged in the mid 1980's as a useful tool to study the geometry of a given variety $X \subset \mathbb{P}^N$ as soon as one has a good knowledge of the hyperplane sections $Y = X \cap H$.

Let us briefly recall their definition and notation in the case of curves.

Notation 1.1. Let C be a smooth irreducible curve and let L, M be two line bundles on C . We denote by $\mu_{L,M} : H^0(L) \otimes H^0(M) \rightarrow H^0(L \otimes M)$ the multiplication map of sections and by $R(L, M) = \text{Ker } \mu_{L,M}$. The Gaussian map associated to L and M will be denoted by

$$\Phi_{L,M} : R(L, M) \rightarrow H^0(\omega_C \otimes L \otimes M).$$

This map can be defined locally by $\Phi_{L,M}(s \otimes t) = sdt - tds$ (see [Wa]).

Perhaps the first important result, proved by Wahl, who introduced Gaussian maps, is that if a smooth curve C lies on a K3 surface, then the Gaussian map $\Phi_{\omega_C, \omega_C}$ cannot be surjective. On the other hand, as it was proved by Ciliberto, Harris and Miranda [CHM], this map $\Phi_{\omega_C, \omega_C}$ is surjective on a curve C with general moduli of genus 10 or at least 12. The link with the study of higher dimensional varieties was provided, around the same period, by Zak, who proved the following result ([Za] - see also [Bd], [Lv]):

If $Y \subset \mathbb{P}^r$ is a smooth variety of codimension at least two with normal bundle N_{Y/\mathbb{P}^r} and $h^0(N_{Y/\mathbb{P}^r}(-1)) \leq r + 1$, then the only variety $X \subset \mathbb{P}^{r+1}$ that has Y as hyperplane section is a cone over Y .

Now the point is that, if Y is a curve, we have the formula

$$h^0(N_{Y/\mathbb{P}^r}(-1)) = r + 1 + \text{cork } \Phi_{H_Y, \omega_Y}$$

where H_Y is the hyperplane bundle of Y .

On the other hand, if Y is not a curve one can take successive hyperplane sections of Y . For example, when $X \subset \mathbb{P}^{r+1}$ is a smooth anticanonically embedded Fano threefold with

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general hyperplane section the K3 surface Y , in [CLM1], Ciliberto, the second author and Miranda were able to compute $h^0(N_{Y/\mathbb{P}^r}(-1))$ by calculating the coranks of Φ_{H_C, ω_C} for the general curve section C of Y . This then led to recover in [CLM1] and [CLM2], in a very simple way, a good part of the classification of smooth Fano threefolds [I1], [I2] and of varieties with canonical curve section [M].

To study other threefolds by means of Zak's theorem, in many cases it is not enough to get down to curve sections and one needs to bound the cohomology of the normal bundle of surfaces. In [KLM] the following general result was proved:

Proposition 1.2. *Let $Y \subset \mathbb{P}^r$ be a smooth irreducible linearly normal surface and let H be its hyperplane bundle. Assume there is a base-point free and big line bundle D_0 on Y with $H^1(H - D_0) = 0$ and such that the general element $D \in |D_0|$ is not rational and satisfies*

- (i) *the Gaussian map $\Phi_{H_D, \omega_D(D)}$ is surjective;*
- (ii) *the multiplication maps μ_{V_D, ω_D} and $\mu_{V_D, \omega_D(D)}$ are surjective, where $V_D := \text{Im}\{H^0(H - D) \rightarrow H^0((H - D)|_D)\}$.*

Then

$$h^0(N_{Y/\mathbb{P}^r}(-1)) \leq r + 1 + \text{cork } \Phi_{H_D, \omega_D}.$$

The application of the above proposition clearly points in the following direction: If one wants to study, with Gaussian maps methods, the existence of threefolds $X \subset \mathbb{P}^{r+1}$ with given hyperplane section Y , one has to know about the surjectivity of Gaussian maps of type Φ_{M, ω_C} for curves $C \subset Y$ that are general in their linear system.

In the present article we do this in the case of Enriques surfaces.

This is applied in [KLM] to prove the (sectional) genus bound $g \leq 17$ for threefolds $X \subset \mathbb{P}^{r+1}$ whose general hyperplane section Y is an Enriques surface.

We prove

Theorem.

Let S be an Enriques surface and let L be a base-point free line bundle on S with $L^2 \geq 4$. Let C be a general smooth curve in $|L|$ and let M be a line bundle on C . Then the Gaussian map Φ_{M, ω_C} is surjective if one of the hypotheses below is satisfied:

- (i) $L^2 = 4$ and $h^0(4L|_C - M) = 0$;
- (ii) $L^2 = 6$ and $h^0((3L + K_S)|_C - M) = 0$;
- (iii) $L^2 \geq 8$ and $h^0(2L|_C - M) = 0$;
- (iv) $L^2 \geq 12$ and $h^0(2L|_C - M) = 1$;
- (v) $H^1(M) = 0$, $\deg(M) \geq \frac{1}{2}L^2 + 2 \geq 6$ and $h^0(2L|_C - M) \leq \text{Cliff}(C) - 2$.

The proof of this theorem will be accomplished essentially in two steps. We will first prove, in Section 2, some general facts about Gaussian maps, by generalizing some known results. Then, in the second step, in Section 5, we will deal with the specific problem of Gaussian maps for curves on Enriques surfaces. As it turns out, the most difficult point will be to show that a tetragonal curve of genus $g \geq 7$ lying on an Enriques surface and general in its linear system, in its canonical embedding, can never be a quadric section of a surface of degree $g - 1$ in \mathbb{P}^{g-1} .

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2. BASIC RESULTS ON GAUSSIAN MAPS

We briefly recall the definition, notation and some properties of gonality and Clifford index of curves.

Definition 2.1. Let X be a smooth surface. We will denote by \sim (respectively \equiv) the linear (respectively numerical) equivalence of divisors (or line bundles) on X . We will say that a line bundle L is **primitive** if $L \equiv kL'$ for some line bundle L' and some integer k , implies $k = \pm 1$.

Definition 2.2. Let C be a smooth irreducible curve of genus $g \geq 2$. We denote by g_d^r a linear system of dimension r and degree d on C and say that C is k -**gonal** (and that k is its **gonality**) if C possesses a g_k^1 but no g_{k-1}^1 . In particular, we call a 2-gonal curve **hyperelliptic**, a 3-gonal curve **trigonal** and a 4-gonal curve **tetragonal**. We denote by $\text{gon}(C)$ the gonality of C .

Definition 2.3. Let C be a smooth irreducible curve of genus $g \geq 4$ and let A be a line bundle on C . The **Clifford index** of A is the integer

$$\text{Cliff}(A) := \deg A - 2(h^0(A) - 1).$$

The **Clifford index** of C is

$$\text{Cliff}(C) := \min\{\text{Cliff}(A) : h^0(A) \geq 2, h^1(A) \geq 2\}.$$

We say that a line bundle A on C **contributes to the Clifford index of C** if $h^0(A) \geq 2, h^1(A) \geq 2$.

2.1. Preliminaries on Gaussian maps. We recall some well-known facts about Gaussian maps.

Proposition 2.4. [Wa, Prop.1.10] Let C be a smooth irreducible nonhyperelliptic curve of genus $g \geq 3$, let $C \subset \mathbb{P}^{g-1}$ be its canonical embedding and let M be a line bundle on C . We have two exact sequences

$$(1) \quad 0 \longrightarrow \text{Coker } \mu_{M, \omega_C} \longrightarrow H^1(\Omega_{\mathbb{P}^{g-1}|C}^1 \otimes \omega_C \otimes M) \longrightarrow H^1(M)^{\oplus g} \longrightarrow H^1(\omega_C \otimes M) \longrightarrow 0$$

and

$$(2) \quad 0 \longrightarrow \text{Coker } \Phi_{M, \omega_C} \longrightarrow H^1(N_{C/\mathbb{P}^{g-1}}^* \otimes \omega_C \otimes M) \longrightarrow H^1(\Omega_{\mathbb{P}^{g-1}|C}^1 \otimes \omega_C \otimes M) \longrightarrow \\ \longrightarrow H^1(\omega_C^2 \otimes M) \longrightarrow 0.$$

In particular

- (a) if $H^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1}) = 0$ then Φ_{M, ω_C} is surjective;
- (b) if $H^1(M) = 0$ and μ_{M, ω_C} is surjective then $\text{cork } \Phi_{M, \omega_C} = h^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1})$.

In the sequel we will collect some results about Gaussian maps of type Φ_{M, ω_C} for curves C of low genus or low gonality or with Clifford index higher than $h^0(2K_C - M) + 2$.

We start with an elementary but useful fact.

Lemma 2.5. For $a \geq 2$, let Q_1, \dots, Q_a be linearly independent homogeneous polynomials of degree 2 in X_0, \dots, X_r . Suppose that the relations among Q_1, \dots, Q_a are generated by $R_i = [R_{i1}, \dots, R_{ia}]$, for $1 \leq i \leq b$. If $(c_1, \dots, c_a) \in \mathbb{C}^a - \{0\}$ then there exists an i such that

$$\sum_{j=1}^a c_j R_{ij} \neq 0.$$

Proof. Suppose to the contrary that $\sum_{j=1}^a c_j R_{ij} = 0$ for every i with $1 \leq i \leq b$.

Without loss of generality assume that $c_1 \neq 0$, so that

$$(3) \quad R_{i1} = - \sum_{j=2}^a c_1^{-1} c_j R_{ij}, \quad 1 \leq i \leq b.$$

Claim 2.6. *Set $Q'_1 = Q_1, Q'_j = Q_j - c_1^{-1} c_j Q_1$ for $2 \leq j \leq a$. Then*

- (i) Q'_1, \dots, Q'_a are linearly independent;
- (ii) the relations among Q'_1, \dots, Q'_a are generated by $S_i = [0, R_{i2}, \dots, R_{ia}]$, for $1 \leq i \leq b$.

Proof. Consider a relation $\sum_{j=1}^a R'_j Q'_j = 0$, where the R'_j 's are polynomials. Then

$$R'_1 Q_1 + \sum_{j=2}^a R'_j (Q_j - c_1^{-1} c_j Q_1) = 0, \text{ whence}$$

$$(4) \quad (R'_1 - \sum_{j=2}^a c_1^{-1} c_j R'_j) Q_1 + \sum_{j=2}^a R'_j Q_j = 0.$$

If all R'_j 's are complex numbers we get $R'_j = 0$ for all j , proving (i).

To see (ii), by (4) and the hypothesis of the lemma we deduce that there are polynomials d_j such that

$$[R'_1 - \sum_{j=2}^a c_1^{-1} c_j R'_j, R'_2, \dots, R'_a] = \sum_{i=1}^b d_i R_i = [\sum_{i=1}^b d_i R_{i1}, \sum_{i=1}^b d_i R_{i2}, \dots, \sum_{i=1}^b d_i R_{ia}]$$

whence $R'_j = \sum_{i=1}^b d_i R_{ij}$ for $2 \leq j \leq a$ and

$$R'_1 = \sum_{j=2}^a c_1^{-1} c_j R'_j + \sum_{i=1}^b d_i R_{i1} = \sum_{i=1}^b d_i (\sum_{j=2}^a c_1^{-1} c_j R_{ij} + R_{i1}) = 0$$

by (3). Now

$$\sum_{i=1}^b d_i S_i = [0, \sum_{i=1}^b d_i R_{i2}, \dots, \sum_{i=1}^b d_i R_{ia}] = [R'_1, R'_2, \dots, R'_a].$$

□

Conclusion of the proof of Lemma 2.5. Consider the Koszul relation $[Q'_2, -Q'_1, 0, \dots, 0]$ among Q'_1, \dots, Q'_a . By the claim there are polynomials d_i such that $\sum_{i=1}^b d_i S_i = [Q'_2, -Q'_1, 0, \dots, 0]$, giving the contradiction $Q'_2 = 0$. □

In many cases, to compute the corank of Gaussian maps, or, as in Proposition 2.4, to compute a suitable cohomology group involving the normal bundle, it is quite convenient to know some surface containing the given curve. The result below will help to compute the cohomology of the normal bundle with the help of the surface.

Lemma 2.7. *Let $Y \subset \mathbb{P}^r$ be an integral subvariety that is scheme-theoretically intersection of quadrics and let $X \subset Y$ be a smooth irreducible nondegenerate subvariety. Let $L = \mathcal{O}_Y(1)$ and M a line bundle on X . Suppose that either*

- (i) $h^0(2L|_X - M) = 0$ or

- (ii) $h^0(2L|_X - M) = 1$ and the relations among the quadrics cutting out Y are generated by linear ones.

Let $\mathcal{F}_{X,Y} = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^r}}(\mathcal{J}_{Y/\mathbb{P}^r}, \mathcal{O}_X)$. Then $H^0(\mathcal{F}_{X,Y} \otimes M^{-1}) = 0$.

Remark 2.8. When Y is smooth we have that $\mathcal{F}_{X,Y} = N_{Y/\mathbb{P}^r}|_X$. The fact that $Y \subset \mathbb{P}^r$ is scheme-theoretically intersection of quadrics certainly holds if Y satisfies property N_1 , that is Y is projectively normal and its homogeneous ideal is generated by quadrics ([L1, Def.1.2.5], [Gr]). Also the fact that the relations among the quadrics cutting out Y are generated by linear ones certainly holds if Y satisfies property N_2 , that is Y satisfies property N_1 and the relations among the quadrics generating its homogeneous ideal are generated by linear ones ([L1, Def.1.2.5], [Gr]). The difference, in our case, is that we do not assume Y to be linearly normal.

Proof of Lemma 2.7. Let $\{Q_1, \dots, Q_a\}$ be linearly independent quadrics cutting out Y scheme-theoretically and consider the corresponding beginning of the minimal free resolution of $\mathcal{J}_{Y/\mathbb{P}^r}$:

$$\bigoplus_{i \geq 0} \mathcal{O}_{\mathbb{P}^r}(-3-i)^{\oplus b_i} \longrightarrow \mathcal{O}_{\mathbb{P}^r}(-2)^{\oplus a} \longrightarrow \mathcal{J}_{Y/\mathbb{P}^r} \longrightarrow 0.$$

Applying the left exact functor $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^r}}(-, \mathcal{O}_X)$ we get an exact sequence

$$0 \longrightarrow \mathcal{F}_{X,Y} \longrightarrow \mathcal{O}_X(2)^{\oplus a} \longrightarrow \bigoplus_{i \geq 0} \mathcal{O}_X(3+i)^{\oplus b_i}$$

whence an exact sequence

$$0 \longrightarrow H^0(\mathcal{F}_{X,Y} \otimes M^{-1}) \longrightarrow H^0(2L|_X - M)^{\oplus a} \xrightarrow{\varphi} \bigoplus_{i \geq 0} H^0((3+i)L|_X - M)^{\oplus b_i}.$$

Then $H^0(\mathcal{F}_{X,Y} \otimes M^{-1}) = \text{Ker } \varphi$.

If we are under hypothesis (i), then obviously $H^0(\mathcal{F}_{X,Y} \otimes M^{-1}) = 0$.

If we are under hypothesis (ii), then $b_i = 0$ for $i \geq 1$ and we will prove that $\text{Ker } \varphi = 0$.

To this end let σ be a generator of $H^0(2L|_X - M)$. For $1 \leq i \leq b_0$ let $R_i = [R_{i1}, \dots, R_{ia}]$ be the linear relations generating all relations among Q_1, \dots, Q_a , so that the map φ is given by the matrix $(R_{ij}|_X)$. If $0 \neq (c_1\sigma, \dots, c_a\sigma) \in \text{Ker } \varphi$ then, for every i such that $1 \leq i \leq b_0$,

we have $\sum_{j=1}^a R_{ij}|_X c_j \sigma = 0$ whence $(\sum_{j=1}^a c_j R_{ij})|_X = 0$. As X is nondegenerate and $\sum_{j=1}^a c_j R_{ij}$

is a linear polynomial, we deduce that $\sum_{j=1}^a c_j R_{ij} = 0$ for all i with $1 \leq i \leq b_0$, contradicting

Lemma 2.5. \square

Now the first general result about Gaussian maps.

Proposition 2.9. *Let C be a smooth irreducible nonhyperelliptic curve of genus $g \geq 3$ and let M be a line bundle on C . We have*

- (a) *If $g = 3$ then $\text{cork } \Phi_{M,\omega_C} \geq h^0(4K_C - M) - \text{cork } \mu_{M,\omega_C} - 3h^1(M)$, with equality if $H^0(-M) = 0$.*
- (b) *If $g = 4$ then $\text{cork } \Phi_{M,\omega_C} \geq h^0(2K_C - M) + h^0(3K_C - M) - \text{cork } \mu_{M,\omega_C} - 4h^1(M)$, with equality if $H^0(-M) = 0$.*
- (c) *If $g = 5$ and C is nontrigonal then $\text{cork } \Phi_{M,\omega_C} \geq 3h^0(2K_C - M) - \text{cork } \mu_{M,\omega_C} - 5h^1(M)$, with equality if $H^0(-M) = 0$.*

- (d) Suppose that C is a plane quintic and A is the very ample g_5^2 on C . If $H^0(5A - M) = 0$ then Φ_{M, ω_C} is surjective. If $H^1(M) = 0$ and μ_{M, ω_C} is surjective then $\text{cork } \Phi_{M, \omega_C} \geq h^0(5A - M)$, with equality holding if in addition $h^0(4A - M) \leq 1$.
- (e) Suppose that C is trigonal, $g \geq 5$ and A is a g_3^1 on C . If $h^0(2K_C - M) \leq 1$ and $H^0(3K_C - (g - 4)A - M) = 0$ then Φ_{M, ω_C} is surjective. If $H^1(M) = 0$ and μ_{M, ω_C} is surjective then $\text{cork } \Phi_{M, \omega_C} \geq h^0(3K_C - (g - 4)A - M)$, with equality holding if in addition $h^0(2K_C - M) \leq 1$.

Proof. Assertions (a), (b) and (c) follow easily from Proposition 2.4.

Let us prove (d). In the canonical embedding $C \subset \mathbb{P}^5$ we have that C is contained in the Veronese surface Y and we have an exact sequence

$$(5) \quad 0 \longrightarrow N_{C/Y} \otimes M^{-1} \longrightarrow N_{C/\mathbb{P}^5} \otimes M^{-1} \longrightarrow N_{Y/\mathbb{P}^5}|_C \otimes M^{-1} \longrightarrow 0.$$

Observe that $h^0(N_{C/Y} \otimes M^{-1}) = h^0(5A - M)$. Now if $h^0(5A - M) = 0$ then also $h^0(2K_C - M) = h^0(4A - M) = 0$ and from (5) and Proposition 2.4 (a), we see that to prove (d) we just need to show that $H^0(N_{Y/\mathbb{P}^5}|_C \otimes M^{-1}) = 0$. The latter follows by Lemma 2.7 and Remark 2.8 since, as is well-known, Y satisfies property N_3 .

Now if $H^1(M) = 0$ and μ_{M, ω_C} is surjective, we have that $\text{cork } \Phi_{M, \omega_C} = h^0(N_{C/\mathbb{P}^5} \otimes M^{-1}) \geq h^0(5A - M)$ by Proposition 2.4 (b) and (5). If we also assume that $h^0(4A - M) = h^0(2K_C - M) \leq 1$ then we can apply again Lemma 2.7 and Remark 2.8. We get that $h^0(N_{Y/\mathbb{P}^5}|_C \otimes M^{-1}) = 0$, whence, from (5), that $h^0(N_{C/\mathbb{P}^5} \otimes M^{-1}) = h^0(5A - M)$.

To see (e) recall that, in the canonical embedding $C \subset \mathbb{P}^{g-1}$, we have [S1, 6.1] that $C \in |3H - (g - 4)R|$ on a rational normal surface $Y \subset \mathbb{P}^{g-1}$, where H is its hyperplane bundle and R its ruling. Since, as is well-known, Y satisfies property N_{g-3} , applying, as in case (d), Lemma 2.7 and Proposition 2.4 we get (e). \square

Note that the cases (d), (e) of the above proposition and the corollary below are a slight improvement of [Te, Thm.2.4] (because we also consider the case $h^0(2K_C - M) = 1$).

Corollary 2.10. *Let C be a smooth irreducible curve of genus $g \geq 5$ and let M be a line bundle on C . Then the Gaussian map Φ_{M, ω_C} is surjective if one of the hypotheses below is satisfied:*

- (a) C is a plane quintic and $\deg M \geq 25$, $M \neq 5A$ if equality holds, where A is the very ample g_5^2 on C ;
- (b) C is trigonal and $\deg M \geq \max\{4g - 6, 3g + 6\}$, $M \neq 3K_C - (g - 4)A$ if $g \leq 12$ and $\deg M = 3g + 6$, where A is a g_3^1 on C .

Proof. (a) follows immediately from Proposition 2.9(d) while (b) is a consequence of Proposition 2.9(e) since, if $h^0(2K_C - M) \geq 2$, then $\deg(2K_C - M) \geq 3$, a contradiction. \square

Another easy but useful consequence of the proof of Lemma 2.7 is the following.

Proposition 2.11. *Let C be a smooth irreducible curve of genus $g \geq 5$ and let M be a line bundle on C . Suppose that either*

- (i) $\text{Cliff}(C) = 2$ and $h^0(2K_C - M) = 0$ or
- (ii) $\text{Cliff}(C) \geq 3$ and $h^0(2K_C - M) \leq 1$.

Then Φ_{M, ω_C} is surjective.

Proof. Since $\text{Cliff}(C) \geq 2$, by [V], [S2], the resolution of the ideal sheaf of the canonical embedding $C \subset \mathbb{P}^{g-1}$ starts as

$$\bigoplus_{i \geq 0} \mathcal{O}_{\mathbb{P}^{g-1}}(-3-i)^{\oplus b_i} \longrightarrow \mathcal{O}_{\mathbb{P}^{g-1}}(-2)^{\oplus a} \longrightarrow \mathcal{I}_{C/\mathbb{P}^{g-1}} \longrightarrow 0$$

with $b_i = 0$ for $i \geq 1$ when $\text{Cliff}(C) \geq 3$. Restricting to C and dualizing we get an exact sequence

$$0 \longrightarrow N_{C/\mathbb{P}^{g-1}} \longrightarrow \mathcal{O}_C(2)^{\oplus a} \longrightarrow \bigoplus_{i \geq 0} \mathcal{O}_C(3+i)^{\oplus b_i}$$

whence an exact sequence

$$0 \longrightarrow H^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1}) \longrightarrow H^0(2K_C - M)^{\oplus a} \xrightarrow{\varphi} \bigoplus_{i \geq 0} H^0((3+i)K_C - M)^{\oplus b_i}.$$

As in the proof of Lemma 2.7 we have that $H^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1}) = 0$ under hypothesis (i) and $H^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1}) = \text{Ker } \varphi = 0$ under hypothesis (ii). Therefore we conclude by Proposition 2.4 (a). \square

Using an appropriate generalization of the methods of [BEL, Proof of Thm.2] we can also get surjectivity when $h^0(2K_C - M) \geq 2$.

Proposition 2.12. *Let C be a smooth irreducible curve of genus $g \geq 4$ and let M be a line bundle on C . Suppose there exists an integer $m \geq 1$ and an effective divisor $D = P_1 + \dots + P_m$ such that*

- (i) $H^1(M - 2P_i) = 0$ for $1 \leq i \leq m$;
- (ii) $h^0(D) = 1$ and $h^0(2K_C - M - D) = 0$;
- (iii) $m \leq \text{Cliff}(C) - 2$.

Then Φ_{M, ω_C} is surjective.

Proof. As is well-known we have $\text{Cliff}(C) \leq \lfloor \frac{g-1}{2} \rfloor$, whence $m \leq \text{Cliff}(C) - 2 \leq g - 4$. We start by observing that $K_C - D$ is very ample. In fact, if $K_C - D$ is not very ample, there are two points $Q_1, Q_2 \in C$ such that

$$h^0(K_C - D - Q_1 - Q_2) = h^0(K_C - D) - 1 = g - 2 - m + h^0(D) = g - 1 - m$$

whence $h^1(D + Q_1 + Q_2) = g - 1 - m \geq 3$ and $h^0(D + Q_1 + Q_2) = 2$ by Riemann-Roch. Therefore $D + Q_1 + Q_2$ contributes to the Clifford index of C and we have $\text{Cliff}(C) \leq \text{Cliff}(D + Q_1 + Q_2) = m$, contradicting (iii).

Consider the embedding $C \subset \mathbb{P}H^0(K_C - D) = \mathbb{P}^r$, where $r = g - 1 - m$. We claim that, in the latter embedding, C has no trisecant lines. As a matter of fact if there exist three points $Q_1, Q_2, Q_3 \in C$ such that their linear span $\langle Q_1, Q_2, Q_3 \rangle$ is a line, we have that

$$\begin{aligned} 1 &= \dim \langle Q_1, Q_2, Q_3 \rangle = h^0(K_C - D) - 1 - h^0(K_C - D - Q_1 - Q_2 - Q_3) = \\ &= g - 1 - m - h^0(K_C - D - Q_1 - Q_2 - Q_3) \end{aligned}$$

whence $h^1(D + Q_1 + Q_2 + Q_3) = g - 2 - m \geq 2$ and again $h^0(D + Q_1 + Q_2 + Q_3) = 2$. Therefore $D + Q_1 + Q_2 + Q_3$ contributes to the Clifford index of C and we get $\text{Cliff}(C) \leq \text{Cliff}(D + Q_1 + Q_2 + Q_3) = m + 1$, contradicting (iii).

Note further that by (ii) and (iii) we have

$$\deg(K_C - D) = 2g - 2 - m \geq 2g + 2 - 2h^1(K_C - D) - \text{Cliff}(C)$$

therefore Green-Lazarsfeld's theorem [L1, Prop.2.4.2] gives that C is scheme-theoretically cut out by quadrics in \mathbb{P}^r . Hence we have a surjection

$$\mathcal{O}_C(2D - 2K_C)^{\oplus \alpha} \rightarrow N_{C/\mathbb{P}^r}^* \rightarrow 0.$$

Setting, as in [BEL], $R_L = N_{C/\mathbb{P}^{H^0(L)}}^* \otimes L$ for any very ample line bundle L , we deduce a surjection

$$\mathcal{O}_C(M - K_C + D)^{\oplus \alpha} \rightarrow R_{K_C - D} \otimes M \rightarrow 0.$$

By (ii), we have that $H^1(M - K_C + D) = H^0(2K_C - M - D)^* = 0$ whence

$$H^1(R_{K_C - D} \otimes M) = 0.$$

Now there is an exact sequence [BEL, 2.7], [E, Proof of Thm.5]

$$0 \longrightarrow R_{K_C - D} \otimes M \longrightarrow R_{K_C} \otimes M \longrightarrow \bigoplus_{i=1}^m \mathcal{O}_C(M - 2P_i) \longrightarrow 0$$

and therefore by (i) we deduce that

$$H^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1}) \cong H^1(N_{C/\mathbb{P}^{g-1}}^* \otimes \omega_C \otimes M)^* \cong H^1(R_{K_C} \otimes M)^* = 0.$$

Hence we get the surjectivity of Φ_{M, ω_C} by Proposition 2.4 (a). \square

We will often use the above result in the following simplified version.

Corollary 2.13. *Let C be a smooth irreducible curve of genus $g \geq 4$ and let M be a line bundle on C such that $H^1(M) = 0$ and $\deg(M) \geq g + 1$. Suppose that*

$$h^0(2K_C - M) \leq \text{Cliff}(C) - 2.$$

Then Φ_{M, ω_C} is surjective.

Proof. Let $m = \text{Cliff}(C) - 2$. Then $m \geq 0$ by hypothesis and when $m = 0$ the surjectivity of Φ_{M, ω_C} holds by Proposition 2.11. When $m \geq 1$ choose general points P_1, \dots, P_m of C and apply Proposition 2.12. \square

Corollary 2.14. [Te, Cor.1.7] *Let C be a smooth irreducible curve of genus $g \geq 5$ nontrigonal and not isomorphic to a plane quintic. Let M be a line bundle on C .*

Then the Gaussian map Φ_{M, ω_C} is surjective if $\deg M \geq 4g - 4$ and $M \neq 2K_C$ if equality holds.

Proof. Immediate consequence of Corollary 2.13 or of Proposition 2.11. \square

2.2. Gaussian maps on tetragonal curves. In this subsection we improve Tendian's [Te] results about Gaussian maps on tetragonal curves. Moreover note that, even though the statement in [Te, Thm.2.10] is almost correct, the proof certainly contains a gap (see Remark 2.17).

We start with some generalities on tetragonal curves following again [S1, 6.2].

Definition-Notation 2.15. *Let C be a smooth irreducible tetragonal curve of genus $g \geq 6$ not isomorphic to a plane quintic. Let A be a g_4^1 on C and let $V_A \subset \mathbb{P}^{g-1} = \mathbb{P}H^0(\omega_C)$ be the rational normal scroll spanned by the divisors in $|A|$, H_A the hyperplane bundle and R_A a ruling of V_A . Let \mathcal{E}_A be the rank 3 vector bundle on \mathbb{P}^1 so that V_A is the image of $\mathbb{P}\mathcal{E}_A$ under the morphism given by $|\mathcal{O}_{\mathbb{P}\mathcal{E}_A}(1)|$. Let \tilde{H}_A and \tilde{R}_A be the pull-backs, under this morphism, of H_A and R_A respectively. Then there are two integers $b_{1,A}, b_{2,A}$ such that $b_{1,A} \geq b_{2,A} \geq 0$,*

$b_{1,A} + b_{2,A} = g - 5$ and there are two surfaces $\tilde{Y}_A \sim 2\tilde{H}_A - b_{1,A}\tilde{R}_A$, $\tilde{Z}_A \sim 2\tilde{H}_A - b_{2,A}\tilde{R}_A$ such that, if Y_A, Z_A are their images in \mathbb{P}^{g-1} then $C = Y_A \cap Z_A$. We also define

$$b_2(C) = \min\{b_{2,A}, A \text{ a } g_4^1 \text{ on } C\}.$$

We have

Lemma 2.16. *The surface $Y_A \subset \mathbb{P}^{g-1}$ has degree $g - 1 + b_{2,A}$ and satisfies property N_2 .*

Proof. We set for simplicity $Y = Y_A$, $\tilde{Y} = \tilde{Y}_A$, $V = V_A$, $\mathcal{E} = \mathcal{E}_A$, $H = H_A$, $R = R_A$, $\tilde{H} = \tilde{H}_A$, $\tilde{R} = \tilde{R}_A$, $b_i = b_{i,A}$, $i = 1, 2$. Note that $\tilde{H}^3 = \deg V = g - 3$, $\tilde{R}^2 = 0$ and $\tilde{H}^2 \cdot \tilde{R} = 1$. Let $\tilde{X} \in |\mathcal{O}_{\tilde{Y}}(\tilde{H})|$ be a general curve. Since $|\mathcal{O}_{\tilde{Y}}(\tilde{H})|$ is not composed with a pencil we have that \tilde{X} is irreducible. Moreover \tilde{X} is smooth outside $\tilde{H} \cap \text{Sing}(\tilde{Y})$, whence \tilde{X} is also reduced.

Let $\mathcal{L} = \mathcal{O}_{\tilde{X}}(\tilde{H})$, $X = Y \cap H$, so that $\varphi_{\mathcal{L}}(\tilde{X}) = X$. We will first prove that X satisfies property N_2 .

To this end by [BF, Thm.A] it is enough to show that

$$(6) \quad \deg X \geq 2p_a(X) + 3.$$

Taking intersections in $\mathbb{P}\mathcal{E}$ we have

$$(7) \quad \deg X = \deg \mathcal{L} = \tilde{H}^2 \cdot \tilde{Y} = \tilde{H}^2 \cdot (2\tilde{H} - b_1\tilde{R}) = 2g - 6 - b_1 = g - 1 + b_2.$$

On the other hand, using the cohomology of the scroll, we get

$$\begin{aligned} p_a(X) &= 1 - \chi(\mathcal{O}_X) = 1 - \chi(\mathcal{O}_Y) + \chi(\mathcal{O}_Y(-1)) = \\ &= 1 - \chi(\mathcal{O}_V) + \chi(\mathcal{O}_V(-2H + b_1R)) + \chi(\mathcal{O}_V(-1)) - \chi(\mathcal{O}_V(-3H + b_1R)) = \\ &= g - 4 - b_1. \end{aligned}$$

Now $2p_a(X) + 3 = 2g - 5 - 2b_1 \leq 2g - 6 - b_1$ if and only if $b_1 \geq 1$. The latter holds because $b_1 \geq b_2 \geq 0$ and $g \geq 6$. Therefore (6) is proved.

Again using the cohomology of the scroll it is easy to prove that $H^1(\mathcal{J}_{Y/\mathbb{P}^{g-1}}(j)) = 0$ for every $j \in \mathbb{Z}$ and that $H^1(\mathcal{O}_Y(j)) = 0$ for every $j \geq 0$. Applying [Gr, Thm.2.a.15 and Thm.3.b.7] (that hold for any scheme) we deduce that Y satisfies property N_2 since $Y \cap H$ does. \square

Remark 2.17. In Tendian's paper it is assumed that a general hyperplane section $Y \cap H$ is smooth, but in fact it can be singular [S1, 6.5] when the g_4^1 exhibits C as a double cover of an elliptic or hyperelliptic curve.

Proposition 2.18. *Let C be a smooth irreducible tetragonal curve of genus $g \geq 6$ not isomorphic to a plane quintic. Let A be a g_4^1 , set $b_2 = b_{2,A}$ and let M be a line bundle on C . We have*

- (i) *If $h^0(2K_C - M) \leq 1$ and $h^0(2K_C - M - b_2A) = 0$, then Φ_{M, ω_C} is surjective;*
- (ii) *If $H^1(M) = 0$ and μ_{M, ω_C} is surjective, then $\text{cork } \Phi_{M, \omega_C} \geq h^0(2K_C - M - b_2A)$, with equality holding if $h^0(2K_C - M) \leq 1$.*

Proof. Let Y be the surface arising in the scroll defined by A and set, as in Lemma 2.7, $\mathcal{F}_{C,Y} = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^{g-1}}}(\mathcal{J}_{Y/\mathbb{P}^{g-1}}, \mathcal{O}_C)$. Applying the left exact functor $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^{g-1}}}(-, \mathcal{O}_C)$ to the exact sequence

$$0 \longrightarrow \mathcal{J}_{Y/\mathbb{P}^{g-1}} \longrightarrow \mathcal{J}_{C/\mathbb{P}^{g-1}} \longrightarrow \mathcal{J}_{C/Y} \longrightarrow 0$$

we get an exact sequence

$$(8) \quad 0 \longrightarrow N_{C/Y} \otimes M^{-1} \longrightarrow N_{C/\mathbb{P}^{g-1}} \otimes M^{-1} \longrightarrow \mathcal{F}_{C,Y} \otimes M^{-1}.$$

Observe that $h^0(N_{C/Y} \otimes M^{-1}) = h^0(2K_C - M - b_2A)$. Now if $h^0(2K_C - M - b_2A) = 0$, from (8) and Proposition 2.4 (a), we see that to prove (i) we just need to show that

$$(9) \quad \text{if } h^0(2K_C - M) \leq 1 \text{ then } H^0(\mathcal{F}_{C,Y} \otimes M^{-1}) = 0.$$

On the other hand, under the hypotheses in (ii), we have that $\text{cork } \Phi_{M,\omega_C} = h^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1})$ by Proposition 2.4 (b). Now from (8) we get that $h^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1}) \geq h^0(2K_C - M - b_2A)$ and to prove equality we need again to prove (9).

To conclude we just note that (9) holds by Lemmas 2.7 and 2.16. \square

3. LINEAR SERIES ON QUADRIC SECTIONS OF SURFACES OF DEGREE $g - 1$ IN \mathbb{P}^{g-1}

In this section we will use some well-known vector bundle methods ([L1], [T]) to study linear series on curves of genus g that are, in their canonical embedding, a quadric section of a surface of degree $g - 1$ in \mathbb{P}^{g-1} . We recall that when the surface is a smooth Del Pezzo the gonality and Clifford index of such curves are known by [P], [Kn1]. Most of the results we prove are probably known, at least in the smooth case, but we include them anyway for completeness' sake.

Lemma 3.1. *Let X be a smooth surface with $-K_X \geq 0$. Let $C \subset X$ be a smooth irreducible curve of genus g and let A be a base-point free g_k^1 on C . Suppose that $2g - 2 - K_X.C - 4k \geq \max\{0, 3 - 4\chi(\mathcal{O}_X)\}$ and, if $h^1(\mathcal{O}_X) \geq 1$, that $h^0(N_{C/X} \otimes A^{-1}) \geq 2h^1(\mathcal{O}_X) + 1$. Then there exist two line bundles L, M on X and a zero-dimensional subscheme $Z \subset X$ such that the following hold:*

- (i) $C \sim M + L$;
- (ii) $k = M.L + \text{length}(Z) \geq M.L \geq L^2 \geq 0$;
- (iii) *there exists an effective divisor D on C of degree $M.L + L^2 - k \geq 0$ such that $A \cong L|_C(-D)$;*
- (iv) *if $L^2 = 0$ then $M.L = k$ and $A \cong L|_C$;*
- (v) *L is base-component free and nontrivial;*
- (vi) *if $C \sim -2K_X$ then $3L^2 + M.L \in 4\mathbb{Z}$.*

Proof. Let $\mathcal{F} = \text{Ker}\{H^0(A) \otimes \mathcal{O}_X \rightarrow A\}$ and $\mathcal{E} = \mathcal{F}^*$. As is well-known ([L1]) \mathcal{E} is a rank two vector bundle sitting in an exact sequence

$$(10) \quad 0 \longrightarrow H^0(A)^* \otimes \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow N_{C/X} \otimes A^{-1} \longrightarrow 0$$

and moreover $c_1(\mathcal{E}) = C$ and $c_2(\mathcal{E}) = k$, so that $\Delta(\mathcal{E}) := c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) = C^2 - 4k = 2g - 2 - K_X.C - 4k \geq 0$. Let H be an ample line bundle on X and suppose that \mathcal{E} is H -stable. Then $h^0(\mathcal{E} \otimes \mathcal{E}^*) = 1$ by [F, Cor.4.8] and $h^2(\mathcal{E} \otimes \mathcal{E}^*) = h^0(\mathcal{E} \otimes \mathcal{E}^*(K_X)) \leq h^0(\mathcal{E} \otimes \mathcal{E}^*) = 1$, therefore $2 \geq h^0(\mathcal{E} \otimes \mathcal{E}^*) + h^2(\mathcal{E} \otimes \mathcal{E}^*) = h^1(\mathcal{E} \otimes \mathcal{E}^*) + \chi(\mathcal{E} \otimes \mathcal{E}^*) \geq 4\chi(\mathcal{O}_X) + \Delta(\mathcal{E}) \geq 3$, a contradiction. Hence \mathcal{E} is not H -stable and if M is the maximal destabilizing subbundle we have an exact sequence

$$(11) \quad 0 \longrightarrow M \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_{Z/X} \otimes L \longrightarrow 0$$

where L is another line bundle on X and Z is a zero-dimensional subscheme of X . Computing Chern classes in (11) we get (i) and the equality in (ii). Since the destabilizing condition reads $(M - L).H \geq 0$ and since $(M - L)^2 = \Delta(\mathcal{E}) + 4\text{length}(Z) \geq 0$, we see that $M - L$ belongs to the closure of the positive cone of X . We now claim that \mathcal{E} is globally generated off a finite set. In fact if $h^1(\mathcal{O}_X) \geq 1$ we have by hypothesis that $h^0(N_{C/X} \otimes A^{-1}) \geq 2h^1(\mathcal{O}_X) + 1$ and the claim follows by (10) since the map $\psi : H^0(\mathcal{E}) \rightarrow H^0(N_{C/X} \otimes A^{-1})$ is nonzero. On the other hand if $h^1(\mathcal{O}_X) = 0$ we have that ψ is surjective, whence, again by (10),

we just need to prove that $h^0(N_{C/X} \otimes A^{-1}) \geq 1$. Since $g \geq 2k + 1 + \frac{1}{2}K_X.C$ we get $\deg(N_{C/X} \otimes A^{-1}) = 2g - 2 - K_X.C - k \geq g$. Therefore $h^0(N_{C/X} \otimes A^{-1}) \geq 1$ by Riemann-Roch and the claim is proved.

Since \mathcal{E} is globally generated off a finite set then so is L . It follows that $L \geq 0$, L is base-component free and $L^2 \geq 0$. Now the signature theorem [BPV, VIII.1] implies that $(M - L).L \geq 0$ thus proving (ii). To see (iii) and (iv) note that if $M.L > 0$ then the nefness of L implies that $H^0(-M) = 0$. On the other hand if $M.L = 0$ then $L^2 = C.L = 0$ whence $L \equiv 0$ by the Hodge index theorem and therefore $C \equiv M$. Then $M.H = C.H > 0$ whence again $H^0(-M) = 0$. Twisting (10) and (11) by $-M$ we deduce that $h^0(L|_C \otimes A^{-1}) \geq h^0(\mathcal{E}(-M)) \geq 1$. This proves (iii) and also (v). Moreover it gives $\deg(L|_C \otimes A^{-1}) \geq 0$, whence, if $L^2 = 0$, we get that $M.L \geq k$. By (ii) it follows that $M.L = k$ and therefore $\deg(L|_C \otimes A^{-1}) = 0$, whence $L|_C \cong A$. This proves (iv).

Finally suppose that $C \sim -2K_X$. We have $\chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}L.(L - K_X)$ whence $2L.(L - K_X)$ is divisible by 4. But $2L.(L - K_X) = 2L^2 + L.C = 3L^2 + M.L$, giving (vi). \square

We now analyze linear series on curves on surfaces of degree r in \mathbb{P}^r . We will use the following

Definition-Notation 3.2. For $1 \leq n \leq 9$ we denote by Σ_n the blow-up of \mathbb{P}^2 at n possibly infinitely near points, by \tilde{H} the strict transform of a line and by G_i the total inverse image of the blown-up points. Let $Q \subset \mathbb{P}^3$ be a quadric cone with vertex V . We denote by $Bl_V Q$ the blow-up of Q along V and by \tilde{H} the strict transform of a plane. Let $C_n \subset \mathbb{P}^n$ be the cone over a smooth elliptic curve in \mathbb{P}^{n-1} and let V be the vertex. We denote by $Bl_V C_n$ the blow-up of C_n along V , by C_0 the inverse image of V and by f the numerical class of a fiber.

Remark 3.3. We recall that by [N, Thm.8] a linearly normal integral surface $Y \subset \mathbb{P}^r$ of degree r is either the anticanonical image of Σ_{9-r} or C_r or the 2-Veronese embedding in \mathbb{P}^8 of an irreducible quadric in \mathbb{P}^3 or the 3-Veronese embedding in \mathbb{P}^9 of \mathbb{P}^2 .

Proposition 3.4. Let X be a surface among Σ_n , $Bl_V Q$ or $Bl_V C_n$ as in Definition 3.2 and let C be a smooth irreducible curve such that, if $X = \Sigma_n$ or $Bl_V Q$ then $C \sim -2K_X$, while if $X = Bl_V C_n$ then $C \equiv -2K_X - 2C_0$. We have:

- (a) if $X = \Sigma_1$ then C has no complete base-point free g_6^1 ;
- (b) if $X = \Sigma_2$ then every complete base-point free g_4^1 on C is $(\tilde{H} - G_i)|_C$, $i = 1, 2$;
- (c) if $X = \Sigma_2$ then every complete base-point free g_6^1 on C is $(2\tilde{H} - G_1 - G_2)|_C - P_1 - P_2$, where P_1, P_2 are two points of C ;
- (d) if $X = \Sigma_3$ then every complete base-point free g_4^1 on C is $(\tilde{H} - G_i)|_C$, $i = 1, 2, 3$;
- (e) if $X = \Sigma_3$ then every complete base-point free g_5^1 on C is either $\tilde{H}|_C - P$ or $(2\tilde{H} - G_1 - G_2 - G_3)|_C - P$, for some point $P \in C$;
- (f) if $X = \Sigma_3$ and A is a complete base-point free g_6^1 on C then either $A \cong (2\tilde{H} - G_i - G_j)|_C - P_1 - P_2$, for $1 \leq i < j \leq 3$ and P_1, P_2 are two points of C or $(-K_X)|_C - A$ is another complete base-point free g_6^1 on C different from $(2\tilde{H} - G_i - G_j)|_C - P_1 - P_2$;
- (g) if $X = Bl_V C_6$ then C has no complete base-point free g_5^1 and every complete base-point free g_4^1 on C is $(f_1 + f_2)|_C$, where f_1, f_2 are two fibers;
- (h) if $X = Bl_V Q$ then C has a unique complete base-point free g_4^1 , namely $f|_C$, where f is the pull-back of a line of the cone Q ;
- (i) if $X = Bl_V Q$ then every complete base-point free g_6^1 on C is $\tilde{H}|_C - P_1 - P_2$, where P_1, P_2 are two points of C ;

- (j) if $X = \text{Bl}_V Q$ then there is no effective divisor $Z \subset C$ such that $f|_C + Z$ is a complete base-point free g_8^2 on C .

Proof. We record, for later use, the following fact on $X = \Sigma_n$. Let \mathcal{L} be a nef line bundle on X with $\mathcal{L} \sim a\tilde{H} - \sum_{i=1}^n b_i G_i$. Then

$$(12) \quad a = \mathcal{L}.\tilde{H} \geq 0, \quad b_i = \mathcal{L}.G_i \geq 0, \quad \mathcal{L}^2 = a^2 - \sum_{i=1}^n b_i^2, \quad \mathcal{L}.(-K_X) = 3a - \sum_{i=1}^n b_i$$

and the Cauchy-Schwartz inequality $(\sum_{i=1}^n b_i)^2 \leq n \sum_{i=1}^n b_i^2$ implies that

$$(13) \quad (3a + \mathcal{L}.K_X)^2 \leq n(a^2 - \mathcal{L}^2).$$

We will now apply Lemma 3.1 to a base-point free g_k^1 indicated in (a)-(i) and we will set $z = \text{length}(Z)$.

(a) We have $K_X^2 = 8$ whence $C^2 = 32, k = 6$ and from (ii) of Lemma 3.1 we deduce that $6 = M.L + z \geq M.L \geq L^2 \geq 0$. Now if $3 \leq L^2 \leq 6$ we have a contradiction by the Hodge index theorem applied to C and L . The same theorem implies, for $L^2 = 2$, that $C \equiv 4L$. But $C \sim 6\tilde{H} - 2G_1$ whence the contradiction $4L.\tilde{H} = C.\tilde{H} = 6$. If $L^2 = 1$ write $L \sim a\tilde{H} - b_1 G_1$. Then $a^2 = b_1^2 + 1$ therefore $a = 1, b_1 = 0$ and $L \sim \tilde{H}$. Then $\deg(L|_C \otimes A^{-1}) = \tilde{H}.C - 6 = 0$, whence $A \cong \tilde{H}|_C$ by (iii) of Lemma 3.1. Therefore we have the contradiction $h^0(A) = 3$. If $L^2 = 0$ by (iv) of Lemma 3.1 we have that $M.L = 6$ whence $3L^2 + M.L = 6$, contradicting (vi) of Lemma 3.1. This proves (a).

(b) We have $K_X^2 = 7, C^2 = 28$ and $k = 4$. By (ii) of Lemma 3.1 and the Hodge index theorem applied to C and L we see that we are left with the case $L^2 = 0$ whence $A \cong L|_C$. By (12), (13) we deduce that $L \sim \tilde{H} - G_i$ for $i = 1, 2$. This proves (b).

(c) We have $K_X^2 = 7$ whence $C^2 = 28$ and $k = 6$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to C and L we get $0 \leq L^2 \leq 2$. The same theorem implies, for $L^2 = 2$, that $z = 0, M.L = 6$. By (iii) of Lemma 3.1 we have that there are two points $P_1, P_2 \in C$ such that $A \cong L|_C - P_1 - P_2$. By (12), (13) we deduce that $L \sim 2\tilde{H} - G_1 - G_2$. If $L^2 = 1$ again by (ii) of Lemma 3.1 and the Hodge index theorem applied to C and L we get that $0 \leq z \leq 1$ and $5 \leq M.L \leq 6$. By (vi) of Lemma 3.1 we have that $M.L = 5$ whence $\deg(L|_C \otimes A^{-1}) = 0$, so that $A \cong L|_C$ by (iii) of Lemma 3.1. By (12), (13) we deduce that $L \sim \tilde{H}$, giving the contradiction $h^0(A) = 3$. If $L^2 = 0$ we have that $M.L = 6$ by (iv) of Lemma 3.1 contradicting (vi) of Lemma 3.1. This proves (c).

(d) We have $K_X^2 = 6$ whence $C^2 = 24$ and $k = 4$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to C and L we get $0 \leq L^2 \leq 1$. The same theorem implies, for $L^2 = 1$, that $z = 0, M.L = 4$, contradicting (vi) of Lemma 3.1. Therefore $L^2 = 0$ and (iv) of Lemma 3.1 implies that $M.L = 4$ and $A \cong L|_C$. By (12), (13) we deduce that $L \sim \tilde{H} - G_i$. This proves (d).

(e) We have $K_X^2 = 6$ whence $C^2 = 24$ and $k = 5$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to C and L we get $L^2 \leq 2$ with equality only when $z = 0, M.L = 5$, contradicting (vi) of Lemma 3.1. When $L^2 = 1$, the same theorem together with (vi) of Lemma 3.1 implies that $z = 0, M.L = 5$, whence $A \cong L|_C - P$ by (iii) of Lemma 3.1. By (12), (13) we deduce that either $L \sim \tilde{H}$ or $L \sim 2\tilde{H} - G_1 - G_2 - G_3$. If $L^2 = 0$ then (iv) of Lemma 3.1 implies that $M.L = 5$, contradicting (vi) of Lemma 3.1. This proves (e).

(f) We have $K_X^2 = 6$ whence $C^2 = 24$ and $k = 6$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to C and L we see, for $3 \leq L^2 \leq 5$, that $z = 0, M.L = 6$, contradicting (vi) of Lemma 3.1. If $L^2 = 2$ by the Hodge index theorem and (vi) of Lemma 3.1 we have that $z = 0, M.L = 6$. By (12), (13) we deduce that $L \sim 2\tilde{H} - G_i - G_j$ for $i \neq j$ and by (iii) of Lemma 3.1 we have that there are two points $P_1, P_2 \in C$ such that $A \cong L|_C - P_1 - P_2$. If $L^2 = 1$ by the Hodge index theorem and (vi) of Lemma 3.1 we have that $z = 1, M.L = 5$. By (12), (13) we deduce that either $L \sim \tilde{H}$ or $L \sim 2\tilde{H} - G_1 - G_2 - G_3$. By (iii) of Lemma 3.1 we have that $A \cong L|_C$, giving the contradiction $h^0(A) = 3$. If $L^2 = 0$ by (iv) of Lemma 3.1 we have that $M.L = 6$ contradicting (vi) of Lemma 3.1. Finally when $L^2 = 6$ the Hodge index theorem applied to C and L implies that $C \equiv 2L$ and $z = 0$. Therefore $L \sim M \sim -K_X$ whence the exact sequence (11) splits since $\text{Ext}^1(\mathcal{O}_X(-K_X), \mathcal{O}_X(-K_X)) = 0$ and we get $\mathcal{E} \cong \mathcal{O}_X(-K_X)^{\oplus 2}$. Therefore $\mathcal{E}(K_X)$ is globally generated and so is $(-K_X)|_C \otimes A^{-1}$ by (10). Moreover again by (10) we get that $(-K_X)|_C \otimes A^{-1}$ is a g_6^1 . Also such a g_6^1 cannot coincide with the other type $(2\tilde{H} - G_i - G_j)|_C - P_1 - P_2$, for otherwise we would have that $(-K_X)|_C \otimes A^{-1} \sim (2\tilde{H} - G_i - G_j)|_C - P_1 - P_2$, whence $A \cong (\tilde{H} - G_k)|_C + P_1 + P_2$ would have two base points. This proves (f).

(g) We have that $X \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-1))$ where $E \subset \mathbb{P}^5$ is a smooth elliptic normal curve. Let C_0 be a section and f be a fiber so that $C_0^2 = -6$ and the intersection form is even. Moreover $C \equiv 2C_0 + 12f$, $C^2 = 24$ and $k = 4, 5$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to C and L we deduce, if $L^2 \geq 2$, that $k = 5$, $L^2 = 2$, $z = 0$ and $M.L = 5$. On the other hand if $L^2 = 0$ we have that $M.L = k$ and $A \cong L|_C$ by (iv) of Lemma 3.1. Let $L \equiv aC_0 + bf$ so that $M \equiv (2-a)C_0 + (12-b)f$ and $L^2 = 2a(b-3a)$. Moreover, by (v) of Lemma 3.1 we have $a = f.L \geq 0$. Now if $L^2 = 2$ we get $a = 1$, $b = 4$ giving the contradiction $M.L = 6$. Therefore $L^2 = 0$ whence either $a = 0$ or $b = 3a$. In the second case we get $k = M.L = 6a$, a contradiction. Therefore $a = 0$ and $k = M.L = 2b$, that is $k = 4$, $b = 2$ and $L \equiv 2f$ as desired. This proves (g).

(h) We have that $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$. Let C_0 be a section and f be a fiber so that $C_0^2 = -2$ and the intersection form is even. Moreover $C \sim 4C_0 + 8f$, $C^2 = 32$ and $k = 4$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to C and L we have a contradiction if $L^2 \geq 2$. Hence $L^2 = 0$, $M.L = 4$ and $A \cong L|_C$ by (iv) of Lemma 3.1. Then we get that either $L \sim f$ or $L \sim C_0 + f$. Since $C_0.C = 0$, this proves (h).

(i) We retain the notation used in (h) except that now $k = 6$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to C and L we deduce, if $L^2 \geq 2$, that $L^2 = 2$, $z = 0$, $M.L = 6$ and $C \equiv 4L$, whence $L \sim C_0 + 2f \sim \tilde{H}$. By (iii) of Lemma 3.1 we have that there are two points $P_1, P_2 \in C$ such that $A \cong \tilde{H}|_C - P_1 - P_2$. When $L^2 = 0$ we get $M.L = 6$ by (iv) of Lemma 3.1, contradicting (vi) of Lemma 3.1. This proves (i).

(j) Again we use the notation in (i). Suppose there is an effective divisor $Z \subset C$ such that $f|_C + Z$ is a complete base-point free g_8^2 on C . By Riemann-Roch we get that

$$h^0((2C_0 + 3f)|_C - Z) = h^0(K_C - f|_C - Z) = 3$$

and the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2C_0 - 5f) \longrightarrow \mathcal{J}_{Z/X}(2C_0 + 3f) \longrightarrow \mathcal{J}_{Z/C}(2C_0 + 3f) \longrightarrow 0$$

gives that also $h^0(\mathcal{J}_{Z/X}(2C_0 + 3f)) = 3$, whence, since $h^0(2C_0 + 3f) = 6$, that Z does not impose independent conditions to $|2C_0 + 3f|$. Now let $Z' \subset Z$ be an effective divisor of

degree 3 and set $Z' + P = Z$. By the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2C_0 - 5f) \longrightarrow \mathcal{J}_{Z'/X}(2C_0 + 3f) \longrightarrow \mathcal{J}_{Z'/C}(2C_0 + 3f) \longrightarrow 0$$

and Riemann-Roch we have

$$\begin{aligned} h^0(\mathcal{J}_{Z'/X}(2C_0 + 3f)) &= h^0(\mathcal{J}_{Z'/C}(2C_0 + 3f)) = h^1(f|_C + Z - P) = \\ &= h^0(f|_C + Z - P) + 1 = 3. \end{aligned}$$

Therefore Z is in special position with respect to $2C_0 + 3f \sim \mathcal{L} + K_X$, where $\mathcal{L} \sim 4C_0 + 7f$. By [R], [GH], [C], [L2] there is a rank 2 vector bundle \mathcal{E} on X sitting in an exact sequence

$$(14) \quad 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_{Z/X} \otimes \mathcal{L} \longrightarrow 0$$

with $c_1(\mathcal{E}) = \mathcal{L}$ and $c_2(\mathcal{E}) = 4$ so that $\Delta(\mathcal{E}) = \mathcal{L}^2 - 16 = 8 > 0$. Therefore \mathcal{E} is Bogomolov unstable and ([Bo], [R]) there are two line bundles A, B on X and a zero-dimensional subscheme $W \subset X$ sitting in an exact sequence

$$(15) \quad 0 \longrightarrow A \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_{W/X} \otimes B \longrightarrow 0.$$

Moreover $\mathcal{L} \sim A + B$, $A.B + \text{length}(W) = 4$, $(A - B)^2 = 8 + 4\text{length}(W)$ and $A - B$ lies in the positive cone of X .

We record for later use two extra properties of A and B .

For every nef line bundle M such that $M^2 \geq 0$ we have:

$$(16) \quad (A - B).M \geq 0;$$

$$(17) \quad A.M \geq 0.$$

To prove (16) and (17) let M be a nef line bundle such that $M^2 \geq 0$. Then $M.H \geq 0$ for every ample H , whence M lies in the closure of the positive cone of X , therefore $(A - B).M \geq 0$ by [BPV, VIII.1]. Now if $A.M < 0$ then also $B.M < 0$ by (16), whence $h^0(A) = h^0(B) = 0$, as M is nef. But this and (15) give $h^0(\mathcal{E}) = 0$, contradicting (14).

Now $(A - B)^2 \geq 8$ and $(A + B)^2 = \mathcal{L}^2 = 24$ therefore

$$(18) \quad A^2 + B^2 \geq 16.$$

Moreover \mathcal{L} lies in the positive cone of X , whence, by [BPV, VIII.1], $(A - B).\mathcal{L} > 0$, that is

$$(19) \quad A^2 > B^2.$$

Now if $A^2 \leq 8$ we deduce by (19) that $B^2 \leq 6$, contradicting (18). Therefore

$$(20) \quad A^2 \geq 10.$$

Suppose that $A \sim aC_0 + a_1f$ so that $B \sim (4 - a)C_0 + (7 - a_1)f$. Intersecting A with the nef divisors $f, C_0 + 2f$ and using (17), we see that $a \geq 0, a_1 \geq 0$, whence $A \geq 0$ and in fact $A > 0$ by (20). Also $a > 0$, for otherwise $A^2 = 0$. Now the exact sequences (14) and (15) twisted by $-A$ give

$$(21) \quad h^0(\mathcal{J}_{Z/X}(B)) \geq h^0(\mathcal{E}(-A)) \geq 1$$

whence also $B > 0$. The nefness of $C_0 + 2f$ then implies $7 - a_1 = B.(C_0 + 2f) \geq 0$, whence $a_1 \leq 7$, while the nefness of f implies that $4 - a = B.f \geq 0$, whence $a \leq 4$. By (16) with $M = C_0 + 2f$ we get $2a_1 - 7 = (A - B).(C_0 + 2f) \geq 0$, whence $a_1 \geq 4$. Finally by (20) we have that $a(a_1 - a) \geq 5$. Therefore we have proved that

$$(22) \quad 1 \leq a \leq 4, \quad 4 \leq a_1 \leq 7, \quad a(a_1 - a) \geq 5.$$

If $a = 1, 2$ we get that $A^2 + B^2 \leq 12$, contradicting (18). Recall now that $C_0 \cap C = \emptyset$ since $C_0.C = 0$. When $a = 3$ we have $A^2 + B^2 = 4a_1 - 6$ whence $a_1 = 6, 7$ by (18). When $a_1 = 7$ we have $B \sim C_0$, whence $B = C_0$. By (21) we deduce the contradiction $Z \subset C_0 \cap C = \emptyset$. When $a_1 = 6$ we have $B \sim C_0 + f$, whence $B = C_0 \cup F$ for some ruling F . As above we have that $Z \cap C_0 = \emptyset$, whence $Z \subset F \cap C$. Since $F.C = 4$ we have that $Z = F \cap C$, whence $Z \sim f|_C$ and therefore $f|_C + Z \sim 2f|_C$ is a complete base-point free g_8^2 on C . This is of course a contradiction since on X we have that $2f|_C$ is a complete base-point free g_8^3 on C . Finally when $a = 4$ we have $B \sim (7 - a_1)f$ whence $a_1 \leq 6$ as $B > 0$. By (22) we get $a_1 = 6$ whence $B \sim f$, therefore again $B = F$ for some ruling F . Hence $Z \subset F \cap C$, giving the same contradiction above. This proves (j). \square

Remark 3.5. Let C be a smooth tetragonal curve of genus 7 such that $\dim W_4^1(C) = 0$ and $\dim W_5^1(C) = 1$ (as in the case $C \sim -2K_X$ on $X = \Sigma_3$). By [ACGH] $W_6^1(C)$ has an irreducible component of dimension at least 3 and whose general element A is a complete g_6^1 on C . Moreover A is base-point free since $\dim W_4^1(C) = 0$ and $\dim W_5^1(C) = 1$. Also the same holds for $K_C - A$ thus proving that, for these curves, there is a family of dimension at least 3 of complete base-point free g_6^1 's whose residual is also base-point free.

4. SOME RESULTS ON ENRIQUES SURFACES

We will use the following well-known

Definition 4.1. Let L be a line bundle on an Enriques surface S such that $L^2 > 0$. Following [CD] we define

$$\phi(L) = \inf\{|F.L| : F \in \text{Pic } S, F^2 = 0, F \not\equiv 0\}.$$

This function has two important properties:

- (i) $\phi(L)^2 \leq L^2$ ([CD, Cor.2.7.1]);
- (ii) If L is nef, then there exists a genus one pencil $|2E|$ such that $E.L = \phi(L)$ ([Co, 2.11] or by [CD, Cor.2.7.1, Prop.2.7.1 and Thm.3.2.1]).

We will often use the

Definition 4.2. Let S be an Enriques surface. A **nodal** curve on S is a smooth rational curve contained in S .

We will now briefly recall some results on line bundles on Enriques surfaces, proved in [KL1] and [KL2], that we will often use.

Lemma 4.3. [KL2, Lemma 2.2] Let $L > 0$ and $\Delta > 0$ be divisors on an Enriques surface S with $L^2 \geq 0$, $\Delta^2 = -2$ and $k := -\Delta.L > 0$. Then there exists an $A > 0$ such that $A^2 = L^2$, $A.\Delta = k$ and $L \sim A + k\Delta$. Moreover if L is primitive then so is A .

Lemma 4.4. [KL2, Lemma 2.3] Let S be an Enriques surface and let L be a line bundle on S such that $L > 0$, $L^2 > 0$. Let $F > 0$ be a divisor on S such that $F^2 = 0$ and $\phi(L) = |F.L|$. Then

- (a) $F.L > 0$;
- (b) if $\alpha > 0$ is such that $(L - \alpha F)^2 \geq 0$, then $L - \alpha F > 0$.

Lemma 4.5. [KL1, Lemma 2.1] Let X be a smooth surface and let $A > 0$ and $B > 0$ be divisors on X such that $A^2 \geq 0$ and $B^2 \geq 0$. Then $A.B \geq 0$ with equality if and only if there exists a primitive divisor $F > 0$ and integers $a \geq 1, b \geq 1$ such that $F^2 = 0$ and $A \equiv aF, B \equiv bF$.

Definition 4.6. *An effective line bundle L on a K3 or Enriques surface is said to be quasi-nef if $L^2 \geq 0$ and $L \cdot \Delta \geq -1$ for every Δ such that $\Delta > 0$ and $\Delta^2 = -2$.*

Theorem 4.7. [KL1, Corollary 2.5] *An effective line bundle L on a K3 or Enriques surface is quasi-nef if and only if $L^2 \geq 0$ and either $h^1(L) = 0$ or $L \equiv nE$ for some $n \geq 2$ and some primitive and nef divisor $E > 0$ with $E^2 = 0$.*

Theorem 4.8. [KL2, Corollary 1] *Let $|L|$ be a base-component free linear system on an Enriques surface S such that $L^2 > 0$ and let $C \in |L|$ be a general curve. Then*

$$\text{gon}(C) = 2\phi(L)$$

unless L is of one of the following types:

- (a) $L^2 = \phi(L)^2$ with $\phi(L) \geq 2$ and even. In these cases $\text{gon}(C) = 2\phi(L) - 2$.
- (b) $L^2 = \phi(L)^2 + \phi(L) - 2$ with $\phi(L) \geq 3$, $L \not\equiv 2D$ for D such that $D^2 = 10$, $\phi(D) = 3$. In these cases $\text{gon}(C) = 2\phi(L) - 1$ except for $\phi(L) = 3, 4$ when $\text{gon}(C) = 2\phi(L) - 2$.
- (c) $(L^2, \phi(L)) = (30, 5), (22, 4), (20, 4), (14, 3), (12, 3)$ and $(6, 2)$. In these cases $\text{gon}(C) = \lfloor \frac{L^2}{4} \rfloor + 2 = 2\phi(L) - 1$.

5. TETRAGONAL CURVES ON ENRIQUES SURFACES AND ON SURFACES OF DEGREE $g - 1$ IN \mathbb{P}^{g-1}

Let C be a smooth irreducible tetragonal curve of genus $g \geq 6$ and let M be a line bundle on C such that $H^1(M) = 0$ and μ_{M, ω_C} is surjective. To have the surjectivity of the Gaussian map Φ_{M, ω_C} , it is necessary, by Proposition 2.18(ii), that $h^0(2K_C - M - b_{2,A}A) = 0$ for every g_4^1 on C . On the other hand when $h^0(2K_C - M) = 1$ we need that $b_{2,A} \geq 1$ for every g_4^1 on C , that is (see 2.15) $b_2(C) \geq 1$, because in this case, by Proposition 2.18(ii), $h^0(2K_C - M - b_{2,A}A) = \text{cork } \Phi_{M, \omega_C}$ is independent of A . As we have seen in 2.15, in the canonical embedding, $C = Y_A \cap Z_A \subset \mathbb{P}^{g-1}$ where Y_A is a surface of degree $g - 1 + b_{2,A}$ by Lemma 2.16. Moreover $b_{2,A} = 0$ if and only if C is a quadric section of Y_A . Therefore saying that $b_2(C) \geq 1$ is equivalent to saying that C , in its canonical embedding, can never be a quadric section of a surface Y_A of degree $g - 1$ in \mathbb{P}^{g-1} .

The present section we will be devoted to proving that tetragonal curves of genus $g \geq 7$, lying on an Enriques surface and general in their linear system, in their canonical embedding, can never be a quadric section of a surface Y_A of degree $g - 1$ in \mathbb{P}^{g-1} . The latter fact will be then used to prove surjectivity of Gaussian maps for such curves in our main theorem.

We start by observing that we cannot do better in genus 6. Let C be a smooth irreducible tetragonal curve of genus 6 and let A be a g_4^1 on C . Now $K_C - A$ is a g_6^2 and has a base point if and only if C is isomorphic to a plane quintic. Therefore if C is not isomorphic to a plane quintic, then it has complete base-point free g_6^2 and either C is bielliptic or the g_6^2 is birational. In the latter case the image of C by the g_6^2 cannot have points of multiplicity higher than 2, therefore C does lie on $X = \Sigma_4$ and is linearly equivalent to $-2K_X$.

Hence we can restrict our attention to curves of genus $g \geq 7$.

We will henceforth let S be an Enriques surface.

Consider a base-point free line bundle L on S with $L^2 \geq 12$ and let $C \in |L|$ be a general curve. By Theorem 4.8 we have that C is not trigonal and moreover C is tetragonal if and only if $\phi(L) = 2$.

Now assume that $\phi(L) = 2$. We have

Theorem 5.1. *Let L be a base-point free line bundle on an Enriques surface with $L^2 \geq 12$ and $\phi(L) = 2$. Then $b_2(C) \geq 1$ for a general curve $C \in |L|$.*

The proof of this theorem will be essentially divided in two parts, namely a careful study of the cases $L^2 = 12, 14$ and 16 and an application of previous results for $L^2 \geq 18$. In both parts we will employ the following

General remark 5.2. Let C be a tetragonal curve of genus g and let A be a g_4^1 on C such that $b_{2,A} = 0$. Then, by 2.15 and Lemma 2.16, in its canonical embedding, C is a quadric section of a surface $Y_A \subset \mathbb{P}^{g-1}$ of degree $g-1$ whence, by Remark 3.3, C is contained in a surface X that is either Σ_{10-g} , or $Bl_V C_{g-1}$, or a smooth quadric in \mathbb{P}^3 or $Bl_V Q$ where Q is a quadric cone in \mathbb{P}^3 , or \mathbb{P}^2 . Also C is either bielliptic (in the case of C_{g-1}) or linearly equivalent to $-2K_X$.

We start with the cases of genus 7, 8 and 9.

5.1. Curves of genus 7. We will need the ensuing

Lemma 5.3. *Let L be a base-point free line bundle on an Enriques surface with $L^2 = 12$ and $\phi(L) = 2$. Let $|2E|$ be a genus one pencil such that $E.L = 2$. Then there exists a primitive divisor E_1 such that $E_1 > 0$, $E_1^2 = 0$, $E + E_1$ is nef, $h^0(E_1) = h^0(E_1 + K_S) = 1$ and one of the following cases occurs:*

- (i) $\phi(L - 2E) = 1$ and $L \sim 3E + 2E_1$, $E.E_1 = 1$;
- (ii) $\phi(L - 2E) = 2$ and $L \sim 3E + E_1$, $E.E_1 = 2$.

Moreover, in case (ii), for any smooth curve $C \in |L|$, we have that $h^0((E_1)|_C) = h^0((E_1 + K_S)|_C) = 2$.

Proof. We have $(L - 3E)^2 = 0$, $E.(L - 3E) = 2$ and by Lemma 4.4 we can write $L \sim 3E + E'_1$ with $E'_1 > 0$, $(E'_1)^2 = 0$ and $E.E'_1 = 2$. Also $1 \leq \phi(L - 2E) \leq \sqrt{(L - 2E)^2} = 2$.

If $\phi(L - 2E) = 2$ we set $E_1 = E'_1$. Then certainly E_1 is primitive and we have $L \sim 3E + E_1$, $E.E_1 = 2$, as in (ii).

If $\phi(L - 2E) = \phi(E + E'_1) = 1$ let $F > 0$ be a divisor such that $F^2 = 0$ and $F.(E + E'_1) = 1$ (F exists by Lemma 4.4). Then necessarily $F.E = 1$, $F.E'_1 = 0$ therefore $E'_1 \equiv 2F$ by Lemma 4.5 and we can set $E_1 = F$. Replacing, if necessary, E with $E + K_S$, we have that E_1 is primitive and $L \sim 3E + 2E_1$, $E.E_1 = 1$, as in (i).

Since E_1 is primitive, to see, in both cases (i) and (ii), that $h^0(E_1) = h^0(E_1 + K_S) = 1$, by [KL1, Cor.2.5], we just need to show that E_1 is quasi-nef. Let $\Delta > 0$ be a divisor such that $\Delta^2 = -2$ and $k := -E_1.\Delta \geq 1$. By [KL2, Lemma2.2] we can write $E_1 \sim A + k\Delta$ for some $A > 0$ primitive with $A^2 = 0$, $A.\Delta = k$. Now $0 \leq L.\Delta = 3E.\Delta + E'_1.\Delta \leq 3E.\Delta - 1$ gives $E.\Delta \geq 1$. From $2 \geq E.E_1 = E.A + kE.\Delta$ we get that either $k = 1$ or $k = 2$, $E.\Delta = 1$ and $E.A = 0$. In the latter case we have that $E \equiv A$ by Lemma 4.5 and this is a contradiction since $A.\Delta = 2$.

Therefore we have proved that E_1 is quasi-nef and if $E_1.\Delta \leq -1$ then $E_1.\Delta = -1$, $E.\Delta \geq 1$. This of course implies that $E + E_1$ is nef.

Suppose now that we are in case (ii), let $F \equiv E_1$ and let $C \in |L|$ be a smooth curve. From the exact sequence

$$0 \longrightarrow F - L \longrightarrow F \longrightarrow F|_C \longrightarrow 0$$

and the fact just proved that $h^0(F) = 1$, $h^1(F) = 0$, we see that $h^0(F|_C) = 1 + h^1(F - L) = 2$ since $F - L \equiv -3E$. \square

The above lemma allows to exclude quickly the bielliptic case.

Remark 5.4. Let L be a base-point free line bundle on an Enriques surface S with $L^2 = 12$ and $\phi(L) = 2$. Let $|2E|$ be a genus one pencil such that $E.L = 2$. Let C be a general curve in $|L|$. If $b_2(C) = 0$ we can certainly say that C is not bielliptic since if A is a complete base-point free g_4^1 on C we have, by Proposition 3.4(g), that $A \sim (f_1 + f_2)|_C$ therefore $|K_C - A| = |(f'_1 + \dots + f'_4)|_C|$ is not birational. On the other hand on the Enriques surface S , if we pick $A = (2E)|_C$, using the notation of Lemma 5.3, we have that either $K_C - A \sim (E + E_1 + K_S)|_C$ or $K_C - A \sim (E + 2E_1 + K_S)|_C$. Since the linear systems $|E + E_1 + K_S|$ and $|E + 2E_1 + K_S|$ define a map whose general fiber is finite by [CD, Thm.4.6.3 and Thm.4.5.1], we get that $|K_C - A|$ is birational for general C since $|L|$ is birational by [CD, Thm.4.6.3 and Prop.4.7.1].

According to the two cases in Lemma 5.3 we will have two propositions.

Proposition 5.5. *Let L be a base-point free line bundle on an Enriques surface with $L^2 = 12$ and $\phi(L) = 2$. Let $|2E|$ be a genus one pencil such that $E.L = 2$ and suppose that $\phi(L - 2E) = 1$.*

Then $b_2(C) \geq 1$ for a general curve $C \in |L|$.

Proof. We use the notation of Lemma 5.3.

First we prove that either $(E + E_1)|_C$ or $(E + E_1 + K_S)|_C$ is a complete base-point free g_5^1 on C .

To this end note that since $(E + E_1)^2 = 2$ and $E + E_1$ is nef by Lemma 5.3, we have by [CD, Prop.3.1.6 and Cor.3.1.4] that either $E + E_1$ or $E + E_1 + K_S$ is base-component free with two base points. Let $B \equiv E + E_1$ be the line bundle that is base-component free. As C is general in $|L|$ we have that $B|_C$ is base-point free. Now the exact sequence

$$0 \longrightarrow B - C \longrightarrow B \longrightarrow B|_C \longrightarrow 0$$

shows that also $B|_C$ is a complete g_5^1 since $B - C \equiv -2E - E_1$ whence $h^1(B - C) = 0$ because $2E + E_1$ is nef by Lemma 5.3.

Now suppose that there exists a line bundle A that is a g_4^1 on C and is such that $b_{2,A} = 0$. By the general remark 5.2 we know that C lies on a surface X (obtained by desingularizing Y_A , if necessary) and either $X = \Sigma_3$, $C \sim -2K_X$ or $X = Bl_V C_6$ and C is bielliptic. As C has a complete base-point free g_5^1 the second case is excluded by Proposition 3.4(g) (or by Remark 5.4). When $X = \Sigma_3$ by Proposition 3.4(e) we know that there is a point $P \in C$ such that either $B|_C \sim \tilde{H}|_C - P$ or $B|_C \sim (2\tilde{H} - G_1 - G_2 - G_3)|_C - P$.

If $B|_C \sim \tilde{H}|_C - P$ then

$$K_C \sim (3\tilde{H} - G_1 - G_2 - G_3)|_C \sim (L + K_S - B)|_C + \tilde{H}|_C - P$$

whence

$$(23) \quad (L + K_S - B)|_C - P \sim (2\tilde{H} - G_1 - G_2 - G_3)|_C \text{ is a } g_6^2 \text{ on } C.$$

If $B|_C \sim (2\tilde{H} - G_1 - G_2 - G_3)|_C - P$ then

$$K_C \sim (3\tilde{H} - G_1 - G_2 - G_3)|_C \sim (L + K_S - B)|_C + (2\tilde{H} - G_1 - G_2 - G_3)|_C - P$$

whence

$$(24) \quad (L + K_S - B)|_C - P \sim \tilde{H}|_C \text{ is a } g_6^2 \text{ on } C.$$

But using the Enriques surface S we have an exact sequence

$$0 \longrightarrow K_S - B \longrightarrow L + K_S - B \longrightarrow (L + K_S - B)|_C \longrightarrow 0$$

and $L + K_S - B \equiv 2E + E_1$, $h^1(K_S - B) = 0$ by Lemma 5.3, whence $(L + K_S - B)|_C$ is a base-point free g_7^2 on C , contradicting (23) and (24). \square

Now the other case.

Proposition 5.6. *Let L be a base-point free line bundle on an Enriques surface S with $L^2 = 12$ and $\phi(L) = 2$. Let $|2E|$ be a genus one pencil such that $E.L = 2$ and suppose that $\phi(L - 2E) = 2$. Then the general curve in $|L|$ possesses no g_6^2 and satisfies $b_2(C) \geq 1$.*

Proof. The proof will be a variant of the method of [KL2, Section4]. By Lemma 5.3 we have $L \sim 3E + E_1$ with $E > 0$, $E_1 > 0$ both primitive, $E^2 = E_1^2 = 0$, E and $E + E_1$ are nef and $E.E_1 = 2$. Let $D = 2E + E_1$ so that $D^2 = 8$, $\phi(D) = 2$, $D.L = 10$ and D is nef, whence base-point free by [CD, Prop.3.1.6, Prop.3.1.4 and Thm.4.4.1].

Now recall that by [CD, Thm.4.6.3 and Thm.4.7.1] the linear system $|D|$ defines a birational morphism $\varphi_D : S \rightarrow \bar{S} \subset \mathbb{P}^4$ onto a surface \bar{S} having some rational double points, corresponding to nodal curves $R \subset S$ such that $D.R = 0$, and two double lines, namely $\varphi_D(E)$ and $\varphi_D(E + K_S)$. More precisely by [Kn2, Prop.3.7] we see that if $Z \subset S$ is any zero-dimensional subscheme of length two not imposing independent conditions to $|D|$ then either $Z \subset E$ or $Z \subset E + K_S$ or any point $x \in \text{Supp}(Z)$ lies on some nodal curve contracted by φ_D . Observe that if $R \subset S$ is a nodal curve contracted by φ_D , then $0 = D.R = E.R + (E + E_1).R$ whence $E.R = E_1.R = 0$ by the nefness of E and of $E + E_1$. This implies that $C.R = 0$, whence that $C \cap R = \emptyset$, for any $C \in |L|_{sm}$. Also, if \bar{S} contains a line different from the two double lines, then this line is image of a nodal curve $\Gamma \subset S$ such that $D.\Gamma = 1$ whence, using again the nefness of $E + E_1$, we have that either $E.\Gamma = 0$, $E_1.\Gamma = 1$ or $E.\Gamma = 1$, $E_1.\Gamma = -1$. This implies that $C.\Gamma = 1, 2$ for any $C \in |L|_{sm}$. In particular, since $C.E = 2$, we find that for each line on \bar{S} its inverse image in S can contain at most two points of any $C \in |L|_{sm}$. Moreover \bar{S} contains finitely many lines, namely the two lines $\varphi_D(E)$, $\varphi_D(E + K_S)$ and the images of the finitely many irreducible curves $\Gamma \subset S$ such that $D.\Gamma = 1$ (these are finitely many since if $D.\Gamma = 1$ we get $\Gamma^2 = -2$).

By Remark 5.4 we know that there is a proper closed subset $B \subset |L|_{sm}$ such that every element in B is bielliptic and by Theorem 4.8 there is another proper closed subset $B_3 \subset |L|_{sm}$ such that every element in B_3 is trigonal or hyperelliptic and any element of $\mathcal{U} := |L|_{sm} - (B \cup B_3)$ is tetragonal. We set B_6^2 for the closed subset of $|L|_{sm}$ whose elements correspond to curves having a g_6^2 .

The goal will be to prove that the open subset $|L|_{sm} - (B \cup B_3 \cup B_6^2)$ is nonempty.

We will therefore suppose that it is empty, so that every $C \in \mathcal{U}$ has a linear series A_C that is a g_6^2 on C .

Since $h^0(D|_C - A_C) = h^0(\omega_C - A_C - (E + K_S)|_C) \geq h^1(A_C) - 2 \geq 1$, we see that there exists an effective divisor T of degree 4 on C such that $T \sim D|_C - A_C$.

Claim 5.7. *For each T as above we have $h^0(\mathcal{J}_{T/S}(D)) = 3$ and $h^0(\mathcal{J}_{T/S}(L)) = 4$.*

Proof. The first part of the claim follows by the exact sequence

$$(25) \quad 0 \longrightarrow \mathcal{O}_S(-E) \longrightarrow \mathcal{J}_{T/S}(D) \longrightarrow \mathcal{J}_{T/C}(D) \longrightarrow 0$$

since then $h^0(\mathcal{J}_{T/S}(D)) = h^0(\mathcal{J}_{T/C}(D)) = h^0(A_C) = 3$.

To see the second part of the claim consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{J}_{T/S} \otimes L \longrightarrow \mathcal{J}_{T/C} \otimes L \longrightarrow 0$$

so that $h^0(\mathcal{J}_{T/S} \otimes L) = 1 + h^0(L|_C - T) = 1 + h^0(A_C + E|_C)$.

We will prove that $h^0(A_C + E|_C) = 3$. Now $h^0(A_C + E|_C) \geq h^0(A_C) = 3$ and we need to exclude that $h^0(A_C + E|_C) \geq 4$.

Assume henceforth that $h^0(A_C + E|_C) \geq 4$.

Since $\deg(A_C + E|_C) = 8$ and $\text{Cliff}(C) = 2$, if $h^0(A_C + E|_C) \geq 4$, we must have $h^0(A_C + E|_C) = 4$, therefore $h^0(\mathcal{J}_{T/S} \otimes L) = 5$. Since $h^0(L) = 7$ we see that there is a zero-dimensional subscheme $Z \subset T$ such that $\text{length}(Z) = 3$ and $h^0(\mathcal{J}_{Z/S} \otimes L) = 5$. We claim that there is a proper subscheme $Z' \subset Z$ such that $\text{length}(Z') = 2$ and $h^0(\mathcal{J}_{Z'/S} \otimes L) \geq 6$. In fact if for every proper subscheme $Z' \subset Z$ with $\text{length}(Z') = 2$ we have $h^0(\mathcal{J}_{Z'/S} \otimes L) = 5$ then Z is in special position with respect to $L + K_S$ and, since $L^2 = 4\text{length}(Z) = 12$, we deduce by [Kn2, Prop.3.7] that there is an effective divisor B such that $Z \subset B$ and $L.B \leq B^2 + 3 \leq 6$. Since $B.L \geq 3$ we get that

$$(26) \quad 3 \leq L.B \leq B^2 + 3 \leq 6$$

whence $0 \leq B^2 \leq 2$. Note that for any $F > 0$ with $F^2 = 0$ we have either $F.L \geq 4$ or $F \equiv E$ (whence $F.L = 2$). Now if $B^2 = 2$ we can write $B \sim F_1 + F_2$ with $F_i > 0$, $F_i^2 = 0$ for $i = 1, 2$ and $F_1.F_2 = 1$. By (26) we have $L.F_1 + L.F_2 = L.B \leq 5$, whence the contradiction $F_1 \equiv E \equiv F_2$. Therefore $B^2 = 0$ and $L.B = 3$ by (26), again a contradiction.

We have therefore proved that there is a proper subscheme $Z' \subset Z \subset C$ such that $\text{length}(Z') = 2$ and $h^0(\mathcal{J}_{Z'/S} \otimes L) \geq 6$, whence $h^0(\mathcal{J}_{Z'/S} \otimes L) = 6$ as L is base-point free and therefore Z' is not separated by the morphism $\varphi_L : S \rightarrow \mathbb{P}^6$. Now recall that by [CD, Thm.4.6.3, Prop.4.7.1 and Cor.1, p.283] φ_L is a birational morphism onto a surface having some rational double points, corresponding to nodal curves $R \subset S$ such that $L.R = 0$, and two double lines, namely $\varphi_L(E)$ and $\varphi_L(E + K_S)$ and that φ_L is an isomorphism outside $E, E + K_S$ and the nodal curves contracted. In particular we deduce that either $Z' = C \cap E$ or $Z' = C \cap E'$. We claim that this implies that either $T \sim (2E)|_C$ or $T \sim (2E + K_S)|_C$.

To see the latter suppose for example that $Z' = C \cap E$ and set $W = T - Z'$ on C . Then $\text{length}(W) = 2$ and $4 = h^0(L|_C - T) = h^0(D|_C + Z' - T) = h^0(D|_C - W)$, whence the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-E) \longrightarrow \mathcal{J}_{W/S} \otimes D \longrightarrow \mathcal{J}_{W/C} \otimes D \longrightarrow 0$$

shows that $h^0(\mathcal{J}_{W/S} \otimes D) = 4$. Therefore W is not separated by the morphism $\varphi_D : S \rightarrow \mathbb{P}^4$. As $C \cap R = \emptyset$, for any nodal curve R contracted by φ_D we have that either $W \sim E|_C$ or $W \sim (E + K_S)|_C$, whence either $T \sim (2E)|_C$ or $T \sim (2E + K_S)|_C$.

Finally since we know that $T \sim D|_C - A_C$ we deduce that either $A_C \sim (E_1)|_C$ or $A_C \sim (E_1 + K_S)|_C$, but this contradicts Lemma 5.3. \square

Continuation of the proof of Proposition 5.6. Consider the following incidence subscheme of $\text{Hilb}^4(S) \times \mathcal{U}$:

$$\mathfrak{J} = \{(T, C) : T \in \text{Hilb}^4(S), C \in \mathcal{U}, T \subset C \text{ and } h^0(D|_C - T) \geq 3\}$$

together with its two projections $\pi : \mathfrak{J} \rightarrow \text{Hilb}^4(S)$ and $p : \mathfrak{J} \rightarrow \mathcal{U}$.

Our assumption that any $C \in \mathcal{U}$ carries a g_6^2 implies, as we have seen, that p is surjective, whence we deduce that \mathfrak{J} has an irreducible component \mathfrak{J}_0 such that $\dim \mathfrak{J}_0 \geq 6$. Since the fibers of π have dimension at most $h^0(\mathcal{J}_{T/S}(L)) - 1 = 3$ by Claim 5.7, we get that $\dim \pi(\mathfrak{J}_0) \geq 3$.

Using $\pi(\mathfrak{J}_0)$ we build up an incidence subscheme of $\pi(\mathfrak{J}_0) \times |D|$:

$$\mathcal{J} = \{(T, D') : T \in \pi(\mathfrak{J}_0), D' \in |D|, T \subset D'\}$$

together with its two projections

$$(27) \quad f : \mathcal{J} \rightarrow |D| \text{ and } h : \mathcal{J} \rightarrow \pi(\mathfrak{J}_0).$$

By (25) and the definition of $\pi(\mathfrak{J}_0)$ we have that h is surjective. Since the fibers of h have dimension $h^0(\mathcal{J}_{T/S}(D)) - 1 = 2$ by Claim 5.7, we find that \mathcal{J} has an irreducible component \mathcal{J}_0 such that $\dim \mathcal{J}_0 \geq 5$.

To show that this fact leads to a contradiction let us return to the morphism $\varphi_D : S \rightarrow \overline{S} \subset \mathbb{P}^4$.

A general hyperplane section $\overline{D} = \overline{S} \cap H \subset \mathbb{P}^3$ is a curve of degree 8 with two nodes, whence of arithmetic genus 7. Consider, for $i = 2, 3$, the exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{S}}(i-1) \longrightarrow \mathcal{O}_{\overline{S}}(i) \longrightarrow \mathcal{O}_{\overline{D}}(i) \longrightarrow 0.$$

Using Riemann-Roch on \overline{D} we get

$$h^0(\mathcal{O}_{\overline{S}}(3)) \leq h^0(\mathcal{O}_{\overline{S}}(2)) + h^0(\mathcal{O}_{\overline{D}}(3)) \leq h^0(\mathcal{O}_{\overline{S}}(1)) + h^0(\mathcal{O}_{\overline{D}}(2)) + h^0(\mathcal{O}_{\overline{D}}(3)) = 33$$

whence $h^0(\mathcal{J}_{\overline{S}/\mathbb{P}^4}(3)) \geq 2$ and therefore there is a plane $\overline{P} \subset \mathbb{P}^4$ such that $\overline{S} \cup \overline{P}$ is a complete intersection of two cubics in \mathbb{P}^4 .

Now every $T \in \pi(\mathfrak{J}_0)$ has three important properties. First of all we know that $T \subset C$ for some $C \in \mathcal{U}$ and $C \cap R = \emptyset$ for every nodal curve R contracted by φ_D , therefore also $T \cap R = \emptyset$ for every nodal curve R contracted by φ_D . Secondly, since $C.E = 2$, we get that $\text{length}(T \cap E) \leq 2$ and $\text{length}(T \cap (E + K_S)) \leq 2$. Thirdly the linear span $l_T := \langle \varphi_D(T) \rangle \subset \mathbb{P}^4$ is a line by Claim 5.7. Moreover let us prove that we cannot have infinitely many elements $T \in \pi(\mathfrak{J}_0)$ such that l_T is the same line. Suppose to the contrary that there is an infinite set $Z \subset \pi(\mathfrak{J}_0)$ and a line $l \subset \mathbb{P}^4$ such that $l_T = l$ for every $T \in Z$. If l is not contained in \overline{S} then it meets \overline{S} in finitely many points, therefore there is a point $P \in l$ and an infinite set $V \subset S$ such that $\varphi_D(x) = P$ for every $x \in V$ and each $x \in V$ lies on some $T \in Z$. Now $V \subset \varphi_D^{-1}(P)$ therefore $\varphi_D^{-1}(P)$, being infinite, must be a nodal curve contracted by φ_D (recall that φ_D is 2 to 1 on E and $E + K_S$) and this is absurd since for any $x \in V$ we have that $x \in T$ for some $T \in Z$ and we know that $T \cap R = \emptyset$ for every nodal curve R contracted by φ_D . Therefore l is contained in \overline{S} and all $T \in Z$ lie in $\varphi_D^{-1}(l) \subset S$ and this is absurd since each T is contained in some $C \in \mathcal{U}$ and we know that $\varphi_D^{-1}(l)$ can contain at most two points of any $C \in \mathcal{U}$.

Since $\dim \pi(\mathfrak{J}_0) \geq 3$ we have that there is a family of lines $l_T := \langle \varphi_D(T) \rangle$ of dimension at least 3 meeting \overline{S} along $\varphi_D(T)$.

Now let $T \in \pi(\mathfrak{J}_0)$ be a general element. We cannot have that $\text{length}(\varphi_D(T)) \geq 4$, else $\varphi_D(T)$ is contained in $l_T \cap F_3$ for every cubic F_3 containing \overline{S} , that is l_T is contained in $\overline{S} \cup \overline{P}$, a contradiction since \overline{S} contains finitely many lines and of course $\overline{P} \cong \mathbb{P}^2$ contains a 2-dimensional family of lines.

Therefore $\text{length}(\varphi_D(T)) \leq 3$ for a general $T \in \pi(\mathfrak{J}_0)$, whence such a T is not mapped isomorphically by φ_D and therefore it does not lie in the open subset $S - E \cup (E + K_S) \cup R_1 \cup \dots \cup R_n$, where R_1, \dots, R_n are the nodal curves contracted by φ_D . This means that for a general $T \in \pi(\mathfrak{J}_0)$ we have that $\varphi_D(T) \cap \text{Sing}(\overline{S}) \neq \emptyset$ and therefore also $\varphi_D(T) \cap \text{Sing}(\overline{D}_0) \neq \emptyset$ for any D_0 containing T .

Now consider the map $f_0 := f|_{\mathcal{J}_0} : \mathcal{J}_0 \rightarrow |D|$ from (27). Certainly f_0 cannot be surjective, for otherwise a general hyperplane section \overline{D}_0 of \overline{S} would have infinitely many lines l_T passing through a fixed node of \overline{D}_0 . Hence the projection of \overline{D}_0 from that node would give either a 2 to 1 map of D_0 onto a singular (since \overline{D}_0 has two nodes) plane cubic, whence

D_0 would be hyperelliptic, or a 3 to 1 map onto a conic, whence D_0 would be trigonal. Therefore $\text{gon}(D_0) \leq 3$ for a general $D_0 \in |D|$, contradicting Theorem 4.8.

Hence $\dim f_0(\mathcal{J}_0) \leq 3$ and let $D_0 \in f_0(\mathcal{J}_0)$ be a general element. If $\dim f_0(\mathcal{J}_0) \leq 2$ we get that $\dim f_0^{-1}(D_0) \geq 3$, whence a general element $(T, D_0) \in f_0^{-1}(D_0)$ is such that at least three points of T are general on D_0 . But this is a contradiction since these points give rise to three general points of $\overline{D_0}$ that span a line.

Therefore $\dim f_0(\mathcal{J}_0) = 3$. We will first prove that this implies that D_0 is reducible.

Suppose that D_0 is irreducible. Note that both $\overline{D_0}$ and D_0 are reduced, for otherwise we would have that $D_0 \sim m\Delta$ for some $\Delta > 0$ and some $m \geq 2$, but then $D_0^2 = 8$ implies $m = 2$, whence that E_1 is 2-divisible, a contradiction. Since $\dim \mathcal{J}_0 \geq 5$ we have that $\dim f_0^{-1}(D_0) \geq 2$ and for each $(T, D_0) \in f_0^{-1}(D_0)$ we know that $l_T = \langle \varphi_D(T) \rangle \subset \mathbb{P}^3$ is a line. Moreover we showed above that we cannot have infinitely many divisors T 's such that $l_T \subset \mathbb{P}^3$ is the same line, therefore $\overline{D_0}$ has a family of dimension at least two of lines l_T meeting $\overline{D_0}$ along $\varphi_D(T)$ and $\text{length}(\varphi_D(T)) \leq 3$ for a general such T . Hence we have a family of dimension at least two of lines meeting $\overline{D_0}$ on a singular point P_0 of $\overline{D_0}$ and meeting it furthermore at two points (possibly coinciding). Also, since $T \cap R = \emptyset$ for every nodal curve $R \subset S$ contracted by φ_D , we see that $\varphi_D(T)$ is not a point. Now a general projection D'_0 of $\overline{D_0}$ in \mathbb{P}^2 has the same property, namely that the general secant line to D'_0 goes through a fixed point (the projection of P_0) and this is absurd since D'_0 is not a line. This proves that D_0 is reducible and we can now assume that

$$f_0(\mathcal{J}_0) \subset \{D' \in |D| : D' \text{ is reducible}\}.$$

To exclude this case we will therefore study the reducible locus of $|D|$. To this end we first prove the following two facts.

Claim 5.8. *There is no decomposition $D \sim A + B$ with $h^0(A) \geq 2$ and $h^0(B) \geq 2$.*

Proof. Suppose such a decomposition exists. Then we get $A.D \geq 2\phi(D) = 4$ and similarly $B.D \geq 4$, whence $A.D = B.D = 4$, since $D^2 = 8$. Let $A \sim F_A + M_A$, $B \sim F_B + M_B$ be the decompositions into base-components and moving parts of $|A|$ and $|B|$. Then $h^0(M_A) \geq 2$ and $h^0(M_B) \geq 2$, whence, as above, $M_A.D = M_B.D = 4$. Now by [CD, Prop.3.1.4] either $M_A \sim 2hE'$ for some genus one pencil $|2E'|$ or $M_A^2 > 0$. In both cases we can write $M_A \sim \sum_{i=1}^n F_i$ with $F_i > 0$, $F_i^2 = 0$ and $n \geq 2$, therefore $4 = M_A.D \geq n\phi(D) = 2n$. Hence $n = 2$ and $M_A \sim 2E$, since for any $F > 0$ with $F^2 = 0$ and $F.D = 2$ we must have $F \equiv E$. Similarly $M_B \sim 2E$ and therefore $2E + E_1 = D \geq 4E$. But then $E_1 \geq 2E$ whence $h^0(E_1) \geq 2$, a contradiction by Lemma 5.3. \square

Claim 5.9. *Let $D \sim \Delta + M$ for some $\Delta > 0$ and $M > 0$ with $M^2 \geq 6$. Then $M^2 = 6$, $\Delta^2 = -2$, $D.\Delta = 0$.*

Proof. By Riemann-Roch we have that $h^0(M) \geq 4$, whence, by Claim 5.8, $h^0(\Delta) = 1$. Hence $\Delta^2 \leq 0$ by Riemann-Roch and $M^2 = (D - \Delta)^2 = 8 + \Delta^2 - 2D.\Delta \geq 6$, so that

$$2D.\Delta \leq 2 + \Delta^2.$$

If $\Delta^2 = 0$ we find the contradiction $2 \geq 2D.\Delta \geq 2\phi(D) = 4$. If $\Delta^2 \leq -2$, by the nefness of D , we find that $0 \geq 2D.\Delta \geq 0$, that is $M^2 = 6$, $\Delta^2 = -2$ and $D.\Delta = 0$. \square

Now the reducible locus:

Claim 5.10. *Let W be an irreducible subvariety of $\{D' \in |D| : D' \text{ is reducible}\}$ such that $\dim W = 3$. Then there is a divisor $G_W > 0$ with $h^0(G_W) = 1$ and such that if*

$M \sim D - G_W$ then $|M|$ is base-component free and every curve $D' \in W$ is $D' = G_W + M'$ for some $M' \in |M|$. Moreover $M^2 = 6$, $G_W^2 = -2$ and $D.G_W = 0$.

Proof. Let D' be an element of W . Since D' is reducible we have that $D' = G + B$ with $G > 0$, $B > 0$ and, by Claim 5.8, we can assume that $h^0(G) = 1$. Since the divisor classes $G > 0$ such that $D - G > 0$ are finitely many, we see that $h^0(B) \geq 4$. Let G' be the base component of $|B|$ and let M be its moving part. Then also $h^0(M) \geq 4$ and $M^2 > 0$, for otherwise we have $M^2 = 0$ whence, by [CD, Prop.3.1.4], we get that $M \sim 2hE'$, with $|2E'|$ a genus one pencil and $h + 1 = h^0(M) \geq 4$, contradicting Claim 5.8, since then $D \sim 2E' + 2(h - 1)E' + G + G'$ and $h^0(2E') = 2$, $h^0(2(h - 1)E' + G + G') \geq 2$. Therefore $1 + \frac{M^2}{2} = h^0(M) \geq 4$, whence $M^2 \geq 6$ and of course $D \sim G + G' + M$ with $G + G' > 0$ and $h^0(G + G') = 1$ by Claim 5.8. By Claim 5.9 we have that $M^2 = 6$, $(G + G')^2 = -2$ and $D.(G + G') = 0$. Therefore $h^0(B) = h^0(M) = 4$.

Since the possible $G + G'$ are finitely many, we get that $\dim W = \dim |M| = 3$. Let G_1, \dots, G_n be the finite set of divisors $G > 0$ such that $D - G > 0$ and let $B_i = D - G_i$ for $i = 1, \dots, n$. We have seen that for every $D' \in W$ there is an $i \in \{1, \dots, n\}$ and a divisor $B' \in |B_i|$ so that $D' = G_i + B'$. Let $\phi_i : |B_i| \rightarrow |D|$ be the natural inclusion defined by $\phi_i(B) = B + G_i$. Then

$$W \subset \bigcup_{i=1}^n \text{Im } \phi_i$$

and since $\text{Im } \phi_i \cong |B_i|$ is a closed subset of $|D|$ and W is irreducible, we deduce that there is some G_W with $h^0(G_W) = 1$, $D' \sim G_W + M$ and every curve $D' \in W$ is $D' = G_W + M'$ for some $M' \in |M|$. Finally the remaining part follows by Claim 5.9. \square

Conclusion of the proof of Proposition 5.6. Recall that $\dim f_0(\mathcal{J}_0) = 3$ and that a general element $D_0 \in f_0(\mathcal{J}_0)$ is reducible. By Claim 5.10, there is a $G > 0$ with $h^0(G) = 1$ and such that if $M \sim D - G$ then $|M|$ is base-component free, $M^2 = 6$, $G^2 = -2$, $D.G = 0$ and every curve $D' \in f_0(\mathcal{J}_0)$ is $D' = G + M'$ for some $M' \in |M|$. Moreover note that every irreducible component of G is a nodal curve contracted by φ_D .

Therefore $D_0 = \bigcup_{i=1}^n R_i \cup M_0$ where the R_i 's are nodal curves contracted by φ_D and M_0 is general in $|M|$. Now M_0 is a smooth irreducible curve by [CD, Prop.3.1.4 and Thm.4.10.2] and $\varphi_D(M_0)$ is a nondegenerate (since $h^0(\sum_{i=1}^n R_i) = 1$) integral curve in \mathbb{P}^3 . On the other hand we know that on D_0 there is a family of dimension at least 2 of divisors T such that $(T, D_0) \in f_0^{-1}(D_0)$ and each T gets mapped to a line l_T by φ_D . Since for each T we have that $T \cap R_i = 0$ for all $i = 1, \dots, n$, we deduce that all these T 's lie in M_0 and this gives a contradiction since then $\varphi_D(M_0)$ would have a two dimensional family of lines l_T as above. We have therefore proved that the general curve $C \in |L|$ possesses no g_6^2 .

To see that it satisfies $b_2(C) \geq 1$ suppose that there exists a line bundle A that is a g_4^1 on C and is such that $b_{2,A} = 0$. By the general remark 5.2 we know that C lies on a surface X (obtained by desingularizing Y_A , if necessary) and either $X = \Sigma_3$, $C \sim -2K_X$ or $X = Bl_V C_6$ and C is bielliptic. But this is clearly a contradiction since in both cases C carries g_6^2 's. \square

5.2. Curves of genus 8.

Proposition 5.11. *Let L be a base-point free line bundle on an Enriques surface with $L^2 = 14$ and $\phi(L) = 2$.*

Then $b_2(C) \geq 1$ for a general curve $C \in |L|$.

We will use the following

Lemma 5.12. *Let L be a base-point free line bundle on an Enriques surface with $L^2 = 14$ and $\phi(L) = 2$. Let $|2E|$ be a genus one pencil such that $E.L = 2$. Then there exists two primitive divisors E_1, E_2 such that $E_i > 0$, $E_i^2 = 0$, $E.E_i = E_1.E_2 = 1$ for $i = 1, 2$,*

$$L \sim 3E + E_1 + E_2$$

and

- (i) $E + E_1$ is nef;
- (ii) either $2E + E_2$ is nef or there exists a nodal curve Γ such that $E_2 \equiv E_1 + \Gamma$, $E.\Gamma = 0$, $E_1.\Gamma = 1$, $E_2.\Gamma = -1$. In particular $2E + E_2$ is quasi-nef.

Moreover let $C \in |L|$ be a general curve. Then

- (iii) either $(E + E_1)|_C$ or $(E + E_1 + K_S)|_C$ is a complete base-point free g_6^1 on C ;
- (iv) $(2E + E_2)|_C$ and $(2E + E_2 + K_S)|_C$ are complete base-point free g_8^2 's on C .

Proof. Using Lemma 4.4 and Lemma 4.5 we can write $L \sim 3E + E_1 + E_2$ with $E_i > 0$ primitive, $E_i^2 = 0$ and $E.E_i = E_1.E_2 = 1$, $i = 1, 2$.

We now claim that we can assume that $E + E_1$ is nef.

Suppose that there is a nodal curve Γ such that $\Gamma.(E + E_1) < 0$. Then $E_1.\Gamma \leq -1 - E.\Gamma \leq -1$ and $k := -E_1.\Gamma \geq 1 + E.\Gamma \geq 1$. By Lemma 4.3, we can write $E_1 \sim A + k\Gamma$ with $A > 0$ primitive with $A^2 = 0$. If $E.\Gamma > 0$ we have that $k \geq 2$ giving the contradiction $1 = E.E_1 = E.A + kE.\Gamma \geq 2$. Therefore $E.\Gamma = 0$ and the nefness of L implies that $E_2.\Gamma > 0$. From $1 = E_2.E_1 = E_2.A + kE_2.\Gamma \geq 1$ we deduce that $k = 1$ and $E_2.A = 0$ whence $E_2 \equiv A$ by Lemma 4.5 and therefore $E_1 \equiv E_2 + \Gamma$. Now if in addition we have that also $E + E_2$ is not nef then the same argument above shows that there is a nodal curve Γ' such that $E_2 \equiv E_1 + \Gamma'$, giving the contradiction $\Gamma + \Gamma' \equiv 0$. Therefore either $E + E_1$ or $E + E_2$ is nef and (i) is proved.

Now let $\Delta > 0$ be such that $\Delta^2 = -2$, $\Delta.(2E + E_2) < 0$. Then $E_2.\Delta \leq -1 - 2E.\Delta \leq -1$ and $k := -E_2.\Delta \geq 1 + 2E.\Delta \geq 1$. By Lemma 4.3, we can write $E_2 \sim A + k\Delta$ with $A > 0$ primitive with $A^2 = 0$. If $E.\Delta > 0$ we have that $k \geq 3$ giving the contradiction $1 = E.E_2 = E.A + kE.\Delta \geq 3$. Therefore $E.\Delta = 0$ and the nefness of L implies that $E_1.\Delta > 0$. From $1 = E_1.E_2 = E_1.A + kE_1.\Delta \geq 1$ we deduce that $k = 1$ and $E_1.A = 0$, whence $E_1 \equiv A$ by Lemma 4.5 and therefore $E_2 \equiv E_1 + \Delta$. Hence $2E + E_2$ is quasi-nef and if it is not nef then we can choose Δ to be a nodal curve. This proves (ii).

To see (iii) note that since $(E + E_1)^2 = 2$ and $E + E_1$ is nef by (i), we have by [CD, Prop.3.1.6 and Cor.3.1.4] that either $E + E_1$ or $E + E_1 + K_S$ is base-component free with two base points. Let $B \equiv E + E_1$ be the line bundle that is base-component free. As C is general in $|L|$ we have that $B|_C$ is base-point free. Now the exact sequence

$$0 \longrightarrow B - C \longrightarrow B \longrightarrow B|_C \longrightarrow 0$$

shows that also $B|_C$ is a complete g_6^1 since $B - C \equiv -2E - E_2$ whence $h^1(B - C) = 0$ by Theorem 4.7 because $2E + E_2$ is quasi-nef.

To see (iv) note that if $2E + E_2$ is nef then it is base-component free with two base points by [CD, Prop.3.1.6, Prop.3.1.4 and Thm.4.4.1] whence $(2E + E_2)|_C$ is base-point free, as C is general. The same argument shows that $(2E + E_1)|_C$ and $(2E + E_1 + K_S)|_C$ are base-point free by (i). Now if $2E + E_2$ is not nef then $2E + E_2 \equiv 2E + E_1 + \Gamma$ by (ii) whence again $(2E + E_2)|_C$ is base-point free, since $\Gamma.C = 0$. Now the exact sequence

$$0 \longrightarrow -E - E_1 \longrightarrow 2E + E_2 \longrightarrow (2E + E_2)|_C \longrightarrow 0$$

shows that also $(2E + E_2)|_C$ is a complete g_8^2 since $h^1(-E - E_1) = 0$ because $E + E_1$ is nef by (i). Similarly we can show the same for $(2E + E_2 + K_S)|_C$. \square

Before proving Proposition 5.11 we use the above lemma to deal with the case of Σ_2 . This is used also in the proof of Proposition 4.17 in [KL2].

Lemma 5.13. *Let L be a base-point free line bundle on an Enriques surface with $L^2 = 14$ and $\phi(L) = 2$. Then the general curve $C \in |L|$ cannot be isomorphic to a curve linearly equivalent to $-2K_X$ on $X = \Sigma_2$.*

Proof. By Lemma 5.12(iii) there is a line bundle B such that $B \equiv E + E_1$ and $B|_C$ is a base-point free complete g_6^1 on C . By Proposition 3.4(c) there are two points $P_1, P_2 \in C$ such that $B|_C \sim (2\tilde{H} - G_1 - G_2)|_C - P_1 - P_2$.

Now

$$K_C \sim (3\tilde{H} - G_1 - G_2)|_C \sim B|_C + P_1 + P_2 + \tilde{H}|_C$$

whence

$$(28) \quad (L + K_S - B)|_C - P_1 - P_2 \sim \tilde{H}|_C \text{ is a } g_6^2 \text{ on } C.$$

On the other hand by Lemma 5.12(iv) we have that $(L + K_S - B)|_C$ is a base-point free g_8^2 on C and this contradicts (28). \square

Proof of Proposition 5.11. Suppose that there exists a line bundle A that is a g_4^1 on C and is such that $b_{2,A} = 0$. By the general remark 5.2 we know that C lies on a surface X (obtained by desingularizing Y_A , if necessary) and either $X = \Sigma_2$, $C \sim -2K_X$ or $X = Bl_V C_7$ and C is bielliptic. The latter case is excluded since, by [KL2, Prop.4.17], C has a unique g_4^1 while the first case was excluded in Lemma 5.13. \square

5.3. Curves of genus 9.

Proposition 5.14. *Let L be a base-point free line bundle on an Enriques surface with $L^2 = 16$ and $\phi(L) = 2$.*

Then $b_2(C) \geq 1$ for a general curve $C \in |L|$.

We will use the following

Lemma 5.15. *Let L be a base-point free line bundle on an Enriques surface with $L^2 = 16$ and $\phi(L) = 2$. Let $|2E|$ be a genus one pencil such that $E.L = 2$. Then there exists a divisor E_1 such that $E_1 > 0$, $E_1^2 = 0$, $E.E_1 = 2$ and*

$$L \sim 4E + E_1.$$

Moreover if $H^1(E_1 + K_S) \neq 0$ there exists a divisor E_2 such that $E_2 > 0$, $E_2^2 = 0$, $E.E_2 = 1$, $E_1 \equiv 2E_2$ and $E + E_2$ is base-component free.

Proof. Since $(L - 4E)^2 = 0$ and $E.(L - 4E) = 2$, by Lemma 4.4 we can write $L \sim 4E + E_1$ with $E_1 > 0$, $E_1^2 = 0$ and $E.E_1 = 2$.

By Theorem 4.7 if $H^1(E_1 + K_S) \neq 0$ then either $E_1 \equiv nE'$ for $n \geq 2$ and some genus one pencil $|2E'|$ or E_1 is not quasi-nef. In the first case we have $2 = nE.E'$ whence $n = 2$, $E.E' = 1$ and we set $E'_2 = E'$. Also $E + E'_2$ is nef in this case.

If E_1 is not quasi-nef there exists a $\Delta > 0$ such that $\Delta^2 = -2$, $\Delta.E_1 \leq -2$. By Lemma 4.3, we can write $E_1 \sim A + k\Delta$ with $A > 0$, $A^2 = 0$, $A.\Delta = k$ and $k = -E_1.\Delta \geq 2$. The nefness of L implies that $E.\Delta > 0$, whence from $2 = E.E_1 = E.A + kE.\Delta \geq 2$ we deduce that $k = 2$, $E.\Delta = 1$ and $E.A = 0$. Hence $A \equiv qE$ for some $q \geq 1$ by Lemma 4.5. Now $2 = A.\Delta = q$ and therefore $E_1 \equiv 2E + 2\Delta$. We now set $E'_2 = E + \Delta$. Let us

prove that $E + E'_2 = 2E + \Delta$ is nef. Let Γ be a nodal curve such that $(2E + \Delta) \cdot \Gamma < 0$. Since now $L \equiv 6E + 2\Delta$ the nefness of L implies that $E \cdot \Gamma > 0$. Now $(2E + \Delta)^2 = 2$ and $(E + \Gamma)^2 \geq 0$ whence $(E + \Gamma) \cdot (2E + \Delta) \geq 1$. But this is a contradiction since $(E + \Gamma) \cdot (2E + \Delta) = 1 + \Gamma \cdot (2E + \Delta) \leq 0$.

Now that $E + E'_2$ is nef we just observe that by [CD, Prop.3.1.6 and Cor.3.1.4] either $E + E'_2$ or $E + E'_2 + K_S$ is base-component free, whence to conclude we choose accordingly $E_2 = E'_2$ or $E_2 = E'_2 + K_S$. \square

Proof of Proposition 5.14. We use the notation of Lemma 5.15.

Suppose that there exists a line bundle A that is a g_4^1 on C and is such that $b_{2,A} = 0$. By the general remark 5.2 we know that C lies on a surface X (obtained by desingularizing Y_A , if necessary) and either $X = \Sigma_1, Bl_V Q$ and $C \sim -2K_X$ or $X = Bl_V C_8$ and C is bielliptic. When $X = \Sigma_1$ or $Bl_V C_8$ we get that C has a complete base-point free g_6^2 and this is excluded by [KL3, Prop.3.5]. The bielliptic case can also be excluded in another way, since, by [KL2, Prop.4.17], C has a unique g_4^1 . Therefore $C \sim -2K_X$ on $X = Bl_V Q$. By Proposition 3.4(h) we have that C has a unique g_4^1 , namely $f|_C$. Hence $(2E)|_C \sim f|_C$ and we deduce that $h^0((4E)|_C) = h^0(2f|_C) = 4$. Now the exact sequence

$$0 \longrightarrow -E_1 \longrightarrow 4E \longrightarrow (4E)|_C \longrightarrow 0$$

shows that $H^1(E_1 + K_S) \neq 0$, since $h^0(4E) = 3$. Therefore there exists a divisor E_2 as in Lemma 5.15.

Let us prove that $(2E + E_2)|_C$ is a complete base-point free g_8^2 on C .

To this end note that since $(2E + E_2)^2 = 4$ and $2E + E_2$ is base-component free with two base points by Lemma 5.15 and [CD, Prop.3.1.6, Prop.3.1.4 and Thm.4.4.1], we have that $(2E + E_2)|_C$ is base-point free. Now the exact sequence

$$0 \longrightarrow 2E + E_2 - C \longrightarrow 2E + E_2 \longrightarrow (2E + E_2)|_C \longrightarrow 0$$

shows that also $(2E + E_2)|_C$ is a complete g_8^2 since $2E + E_2 - C \equiv -2E - E_2$ whence $h^1(2E + E_2 - C) = 0$ because $2E + E_2$ is nef.

Let $Z = E_2 \cap C$. Then $Z \subset C$ is an effective divisor such that $f|_C + Z \sim (2E + E_2)|_C$ is a complete base-point free g_8^2 on C , contradicting Proposition 3.4(j). \square

We can now complete the proof of Theorem 5.1.

Proof of Theorem 5.1. By Lemma 5.3, Propositions 5.5, 5.6, 5.11 and 5.14 we can assume $L^2 \geq 18$.

Let C be a curve as in the theorem, let $g = \frac{L^2}{2} + 1 \geq 10$ be the genus of C and suppose that $b_2(C) = 0$. By the general remark 5.2 either C is bielliptic or $g = 10$ and C is isomorphic to a smooth plane sextic. Now by [KL2, Prop.4.17] we have that C has a unique g_4^1 , therefore it cannot be bielliptic. On the other hand the case of C isomorphic to a smooth plane sextic is excluded in [KL3, Prop.3.1]. Therefore we have a contradiction in all cases and the theorem is proved. \square

6. PROOF OF THE MAIN THEOREM

We proceed with our main result.

Proof. Let C be a curve as in the theorem and let $g = \frac{L^2}{2} + 1 \geq 3$ be its genus.

Under the hypotheses (i) and (ii) the theorem follows immediately from Proposition 2.4, while if hypothesis (v) holds the theorem follows immediately from Corollary 2.13.

Now suppose we are under hypothesis (iii). By Theorem 4.8 and [KL2, Prop.4.15] we have that C is neither trigonal nor isomorphic to a smooth plane quintic, that is $\text{Cliff}(C) \geq 2$. Then the theorem follows by Proposition 2.11.

Finally suppose that hypothesis (iv) holds. Since $L^2 \geq 12$, by [GLM, Thm.1.4] (or by Theorem 4.8) we get that $\text{Cliff}(C) \geq 2$. If $\text{Cliff}(C) \geq 3$ then (iv) follows by Proposition 2.11(ii). If $\text{Cliff}(C) = 2$ then, as is well-known, C is either tetragonal or isomorphic to a smooth plane sextic. But the latter case was excluded in [KL3, Prop.3.1]. Therefore C is tetragonal and $\phi(L) = 2$ by Theorem 4.8. By Theorem 5.1 we have that $b_2(C) \geq 1$. Since $h^0(2K_C - M) = 1$ it follows that $h^0(2K_C - M - b_2A) = 0$ for every line bundle A that is a g_4^1 on C . Therefore the theorem is a consequence of Proposition 2.18(i). \square

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